Gas Dynamics

Course Code: ME-431

CREDIT:03

Course Teacher: Iftekhar Mahmud

TOTAL MARKS:150

Mid Exam Hours 2

CIE MARKS: 90

Semester End Exam Hours 3

SEE MARKS: 60

Course Learning Outcomes (CLOs): After completing this course successfully, the students will be able to-

CLO 1	Defining and applying the concept of Mach number to differentiate between subsonic, sonic, and supersonic flows.
CLO 2	Calculating changes in flow properties through isentropic processes (no friction or heat transfer).
CLO 3	Analyzing oblique shock waves and their interaction
CLO 4	Exploring the effects of high-temperature gas flows and real gas properties
CLO 5	Utilizing computational fluid dynamics (CFD) techniques to simulate complex gas flow
CLO 6	Identifying the governing equations for compressible flow, including conservation of mass, momentum, and energy

SL	Content of Course	Hrs	CLOs
1	One dimensional flow with area change, friction, and heat transfer.	8	CLO1
2	Flow in converging-diverging nozzles; Governing compressible flow equations, Transonic flow; Stationary, detached and moving shocks.	8	CLO3, CLO4
3	Generation of shocks over wedge and its expansion; supersonic and hypersonic flows.	9	CLO2, CLO6
4	Shock interaction in supersonic flows.	9	CLO5, CLO6

Text Book:
1.Fundamentals of Gas Dynamics, 3rd Edition, Robert D. Zucker, Oscar Biblarz.
P. Balachandran, —Fundamentals of Compressible fluid dynamics‖, PHI Learning, New Delhi

Course plan specifying content, CLOs, teaching learning and assessment strategy mapped with CLOs

Week	Topic	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLOs
1	Introduction, Definition of Quality, Methods of control chance causes and assignable causes,	Lecture, discussion, group work	Quiz, Written Exam	CLO1
2	causes,Seven statistisfial tools ,problem solving methodology (PDCA)	Oral Presentation, debate	Assignment, Written, Quiz	CLO1
3	seven management Tools ,SQC Benefits and limitations, Quality function	Video lecture, Field visit	Report writing, Demonstration	CLO1
4	Quality assurance,Quality audit, Quality cost, Quality circle, Team of Quality circle,Benefits	Lecture	Viva, Quiz	CLO1, CLO3
5	Theory of Control Charts, control chart for variables X & R Chart, Standard deviationchart	Project exhibition	Project, Field visit	CLO1
6	Process capability studies Control Chart for attributes, fraction defective and of defective charts chart	Discussion, Video Presentation	Quiz, Written Exam	CLO1
7	sensitivity Control chart for non conformities (p, np chart, c &u charts) problems using SQC tables	Case-based Learning, Demonstration	Assignment, Written, Quiz	CLO1, CLO4
8	Acceptance sampling, Fundamental concepts & terms ,Operation characteristic curves(OC curves) AQL	Lecture, discussion, group work	Report writing, Demonstration	CLO3, CLO4
9	LTPD, AOQL, sampling plans for single ,Double Multiple sampling plan,sequential sampling plan,	Oral Presentation, debate	Viva, Quiz	CLO2
10	Dodge Roming sampling plans Lot by Lotacceptance sampling by Attributes	Video lecture	Project, Field visit	CLO2
11	,AQL system for Lot by Lot sampling,Acceptancesampling by variables.	Lecture	Quiz, Written Exam	CLO3, CLO4
12	Quality policy, Quality planning, designing for quality, Manufacturing for quality	Project exhibition	Assignment, Written, Quiz	CLO3, CLO3
13	quality philosophies by Deming and Jurang, Crosby & Muller,TQM definition,customer focus	Discussion, Video Presentation	Report writing, Demonstration	CLO1
14	Top Management commitment, Team work,	Case-based Learning, Demonstration	Viva, Quiz	CLO1
15	Implementation of TQM,Concepts of Kaizen, 5S,	Lecture, discussion, group work	Project, Field visit	CLO3, CLO4
16	just in time JIT, Taguchi methods, ;	Oral Presentation, debate	Quiz, Written Exam	CLO3, CLO4
17	need for ISO 9000,clausesof ISO 9000 Implementation,Case studies	Video lecture	Assignment, Written, Quiz	CLO3, CLO4

Introduction

Fluid

- Substance capable of flowing.

  Eg: Liquid, gases and vapour

Fluid Statics

- The study of fluid at rest.

Fluid Kinematics

- The study of fluid in motion without considering the pressure.

Dheenathayalan

Introduction

Fluid dynamics

– The study of fluid in motion where pressure force is considered.

Dheenathayalan

Gas Dynamics

Compressible fluid dynamics – Gas dynamics

Gas dynamics – The branch of fluid dynamics which is concerned with the causes and effect arising from the motion of compressible flow.

Gas Dynamics

Application

- Used in steam and gas turbines.
- High speed aerodynamics.
- High speed turbo compressors.
- Jet, rocket and missile propulsion system.
- Transonic, supersonic and hypersonic flows.

Fundamental laws

Steady flow energy equation – first law of thermodynamics

Entropy relations – second law of thermodynamics

Continuity equation – law of conservation of mass

Momentum equation – Newton's second law of motion

Basic Definitions

System

– An arbitrary collection of matter which has a fixed identity.

Surrounding

– Anything outside the system.

Boundary

– Imaginary surface which separate the system from its surroundings

Dheenathayalan

Basic Definitions

Closed system

- There is no mass transfer between the system into the surroundings but energy (or) heat transfer takes place.

Open system

- Both energy and mass transfer takes place from the system into the surroundings.

Isolated system

- if there is no mass transfer and energy transfer to and from the system

Basic Definitions

State

– Each and every condition of the system.

Process

– A change or a serious of change in the state of the system.

Cycle

– The series of processes whose end state are identical.

Basic Definitions

Property

- An observable characteristics of the system.

Intensive property

- The property which is independent on mass of the system.

  Eg. pressure, temperature, density, viscosity

Extensive property

- The property which depends on the mass of the system.

  Eg. Volume, enthalpy, work done

Basic Definitions

Pressure – the normal force per unit area. SI unit N/m^2.

$1\ N/m^2 = 1$ Pascal $=1Pa$

$1\ bar = 10^5\ N/m^2$

1 atm = 1.01325 bar

1 atm = 760 mm of Hg

1 atm = 10.336 m of water column

Temperature – when two system are in contact with each other and are in thermal equilibrium, the property common to both the systems having the same value is called temperature.

$1^oC = 1+273\ K$

Basic Definitions

Density – the mass of the substance per unit volume.

- SI unit Kg/m^3.

Work – It is an energy which is the product of force and the distance travelled in the direction of force.

- it is path function not a property.
- Unit N.m (or) Joule.

Dheenathayalan

Basic Definitions

Specific heat – the amount of heat that required to rise the temperature of unit mass of substance by one degree.

Specific heat capacity at constant volume (C_v)

- if temperature rise occurs at constant volume

Specific heat capacity at constant pressure(C_p)

- if temperature rise occurs at constant pressure

The characteristic gas constant

$$R = C_p - C_v$$

Basic Definitions

The characteristic gas constant

$$R = C_p - C_v$$

Divide throughout by C_p

$$\frac{R}{C_p} = 1 - \frac{C_v}{C_p} = \left(1 - \frac{1}{\gamma}\right)$$

$$\frac{R}{C_p} = \left(\frac{\gamma - 1}{\gamma}\right)$$

$$\boxed{C_p = \frac{\gamma R}{(\gamma - 1)}}$$

Dheenathayalan

Basic Definitions

Adiabatic process

- During a process if there is no heat transfer between the system and the surroundings.
- Rotodynamic machines (or) turbo machines assumed to followed only adiabatic process.

Dheenathayalan

Basic Definitions

Isentropic process – in which there is no change in entropy, such process is a reversible adiabatic process or isentropic process.

This is governed by the following relations:

$$pv^{\gamma} = constant$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} = \left(\frac{v_1}{v_2}\right)^{\gamma-1}$$

$$Tds = dh - vdp = dh - \frac{dp}{\rho}$$

Types of Flow

- ✓ Steady and unsteady flow
- ✓ Uniform and non-uniform flow
- ✓ Laminar and turbulent flow
- ✓ Compressible and incompressible flow
- ✓ Rotational and irroational flow
- ✓ One dimensional, two dimensional and three dimensional flow

Dheenathayalan

Steady & Unsteady flow

- ✓ Steady flow is that type of flow, in which the fluid characteristics like velocity, pressure and density at a point do not change with time.

- ✓ Unsteady flow is that type of flow, in which the fluid characteristics like velocity, pressure and density at a point changes with respect to time.

Uniform & Non - Uniform

- ✓ Uniform flow, in which velocity of fluid particles at all sections are equal.
- ✓ Non – uniform flow, in which velocity of fluid particles at all sections are not equal.

Laminar & Turbulent Flow

- ✓ Laminar flow or stream line flow, the fluid moves in layers and each particle follows a smooth and continuous path.

- ✓ Turbulent flow, the fluid particles move in very irregular path

Dheenathayalan

Compressible & Incompressible Flow

- ✓ Compressible flow in which density of fluid changes from point to point (Gases, Vapours). ($M>0.3$)
- ✓ Incompressible flow in which density of fluid is constant (Liquids). ($M<0.3$)

Mach Number = Flow velocity/Velocity of sound

$$M = \frac{c}{a}$$

Dheenathayalan

Rotational & Irrotational flow

- ✓ Rotational flow in which the fluid particles flowing along stream lines and also rotate about their own axis.

- ✓ Irrotational flow in which the fluid particles flowing along stream lines but don’t rotate about their own axis (without turbulences, whirlpool, vortices, etc.,).

1-D, 2-D & 3-D Flow

- ✓ 1-D flow, in which the flow parameter such as velocity is a function of time and one space co-ordinate (x) only (stream lines - straight line).
- ✓ 2-D flow, in which the flow parameter such as velocity is a function of time and two space co-ordinate (x, y) only
- ✓ 3-D flow, in which the flow parameter such as velocity is a function of time and three mutual perpendicular axis (x, y, z) only.

Energy equation

The first law of thermodynamics state that, when a system execute a cyclic process, the algebraic sum of work transfer is proportional to the algebraic sum of heat transfer.

$$\oint dW \propto \oint dQ$$

$$\oint dW = J \oint dQ$$

When heat and work terms are expressed in the same units

$$\oint dW - \oint dQ = 0$$

Dheenathayalan

Energy equation

The quantities 'dQ' and 'dW' will follow the path function, but the quantity (dQ-dW) does not depands on the path of the process. Therefore the change in quantity (dQ-dW) is a property called Energy (E).

$$dE = dQ - dW$$

$$\int_1^2 dE = \int_1^2 dQ - \int_1^2 dW$$

$$(E_2 - E_1) = Q - W$$

$$Q = W + (E_2 - E_1) \qquad .....(i)$$

Energy equation

In the above equation 'E' may includes kinetic energy, internal energy, gravitational potential energy, strain energy, magnetic energy etc.,

By ignoring magnetic energy and strain energy term 'E' written as

$$E = U + mgZ + \frac{1}{2}mc^2 \qquad \ldots\ldots(ii)$$

The differential form of eqn. *(ii)* is

$$dE = dU + mgZ + md(\frac{1}{2}c^2)$$

Integrating the above eqn.

$$\int_1^2 dE = \int_1^2 dU + mg\int_1^2 dZ + \frac{1}{2}m\int_1^2 d(c^2)$$

Energy equation

$$E_2 - E_1 = (U_2 - U_1) + mg(Z_2 - Z_1) + \frac{1}{2}m(c_2^2 - c_1^2) \qquad .....(iii)$$

Substituting eqn. (iii) in (i)

$$Q = W + (U_2 - U_1) + mg(Z_2 - Z_1) + \frac{1}{2}m(c_2^2 - c_1^2) \qquad .....(iv)$$

Dheenathayalan

Energy equation for a flow process:

A change or a series of changes in an open system is known as "flow process"

Eg:

i. Flow through nozzles, diffusers and duct etc.,

ii. Expansion of steam and gases in turbines

iii. Compression of air and gases in turbo compressor etc.,

In such flow processes the work term (W) includes flow work also

$$W = W_S + (p_2V_2 - p_1V_1) \qquad .....(v)$$

Where, W_s = Shaft work ; $(p_2v_2-p_1v_1)$ = Flow work

Energy equation for a flow process:

Substituting eqn. (iv) in (v)

$$Q = W_S + (p_2V_2 - p_1V_1) + (U_2 - U_1) + mg(Z_2 - Z_1) + \frac{1}{2}m(c_2^2 - c_1^2)$$

$$Q = W_s + (U_2 + p_2v_2) - (U_1 + p_1v_1) + mg(Z_2 - Z_1) + \frac{1}{2}m(c_2^2 - c_1^2)$$

But, we know that H=U+pV

$$Q = W_s + (\mathrm{H}_2 - \mathrm{H}_1) + mg(Z_2 - Z_1) + \frac{1}{2}m(c_2^2 - c_1^2) \qquad .....(vi)$$

$$h_1 + gZ_1 + \frac{1}{2}mc_1^2 + q = h_2 + gZ_2 + \frac{1}{2}mc_2^2 + w_S \qquad .....(vii)$$

Energy equation for a flow process:

$$h_1 + gZ_1 + \frac{1}{2}mc_1^2 + q = h_2 + gZ_2 + \frac{1}{2}mc_2^2 + w_S \qquad .....(vii)$$

Where,

h_1, h_2= Enthalpy of flowing fluid at inlet and outlet

Z_1, Z_2= Datum heads at inlet and outlet

c_1, c_2= Fluid velocity at inlet and outlet

q = Heat flow in the system

w_s = Shaft work in the system

Equation (vii) is a steady flow energy equation per unit Kg mass.

Dheenathayalan

Adiabatic energy equation:

- Compared to other quantities, the change in elevation $g(Z_2-Z_1)$ is negligible in flow problems of gases and vapours.
- In a reversible adiabatic process the heat transfer 'q' is negligibly small and can be ignored.
- Expansion of gases and vapours in nozzles and diffusers are example for this process.
- For this process eqn.(vii) reduced to

$$h_1 + \frac{c_1^2}{2} = h_2 + \frac{c_2^2}{2} \quad \ldots\ldots(viii)$$

$$[w_s = 0]$$

Adiabatic energy transfer and energy transformation:

Adiabatic energy transfer:

shaft work will present in an adiabatic energy transfer process

$$h_1 + \frac{c_1^2}{2} = h_2 + \frac{c_2^2}{2} + w_s \qquad \ldots\ldots(ix)$$

Example:

i. Expansion of gases in turbines

ii. Compression of gases in compressor

Dheenathayalan

Adiabatic energy transfer and energy transformation:

Adiabatic energy transformation:

In adiabatic energy transformation process the shaft work is zero.

$$h_1 + \frac{c_1^2}{2} = h_2 + \frac{c_2^2}{2} \qquad \ldots\ldots(x)$$

Example:

i. Expansion of gases in nozzle

ii. Compression of gases in diffuser

Stagnation state and stagnation properties:

Stagnation state:

- ✓ The state of a fluid attained by isentropically decelerating it to zero velocity at zero elevation is referred to as stagnation state.
- ✓ The properties of fluid at stagnation state are the stagnation properties of the fluid.
- ✓ Eg: stagnation temperature, stagnation pressure, stagnation enthalpy, stagnation density

Dheenathayalan

Stagnation enthalpy [h_o]:

Stagnation enthalpy of a gas or vapour is its enthalpy when it is adiabatically decelerated to zero velocity at zero elevation.

As the definition,

At the initial state $h_1 = h : c_1 = c$

At the final state $h_2 = h_o : c_2 = 0$

Substituting the eqn. (x),

$$h_o = h + \frac{c^2}{2} \qquad \ldots\ldots(xi)$$

Where, h_o=Stagnation enthalpy and h = static enthalpy

Stagnation temperature [T_o]:

Stagnation temperature of a gas or vapour is defined as temperature when it is adiabatically decelerated to zero velocity at zero elevation.

for perfect gas, eqn (xi) can be written as,

$$C_p T_o = C_p T + \frac{c^2}{2}$$

Divide the equation throughout by C_p

$$\therefore T_0 = T + \frac{c^2}{2C_p} \qquad \ldots\ldots (xii)$$

Where, $T_o = Sagnation\ temperature$

$T = Static\ temperature$

$\frac{c^2}{2C_p} = Velocity\ temperature$

Stagnation temperature [T_o]:

$$\frac{T_o}{T} = 1 + \frac{c^2}{2C_pT}$$

$$\left[C_p = \frac{\gamma R}{(\gamma - 1)}\right]$$

$$\frac{T_o}{T} = 1 + \frac{c^2}{2\frac{\gamma}{(\gamma - 1)}RT}$$

$$\left[a = \sqrt{\gamma RT}\right]$$

$$\frac{T_o}{T} = 1 + \frac{(\gamma - 1)}{2} \times \frac{c^2}{a^2}$$

$$\left[M = \frac{c}{a}\right]$$

$$\frac{T_o}{T} = 1 + \frac{(\gamma - 1)}{2}M^2$$

$$\boxed{\frac{T_o}{T} = 1 + \frac{(\gamma - 1)}{2}M^2} \quad .....(xiii)$$

Stagnation pressure[P_o]:

Stagnation temperature is the pressure of gas when it is adiabatically decelerated to zero velocity at zero elevation.

for perfect gas, the adiabatic reaction is

$$\frac{T_o}{T} = \left(\frac{P_o}{P}\right)^{\frac{(\gamma-1)}{\gamma}}$$

$$\frac{P_o}{P} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{(\gamma-1)}}$$

$$\frac{P_o}{P} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{(\gamma-1)}} \quad .....(xiv)$$

Dheenathayalan

Stagnation velocity of sound [a_o]:

we know that the acoustic velocity of sound

$$a = \sqrt{\gamma R T}$$

For the given value of stagnation temperature the stagnation velocity of sound

$$\boxed{a_o = \sqrt{\gamma R T_o}}$$

Dheenathayalan

Stagnation density [ρ_o]:

For the given value of stagnation pressure and temperature the stagnation density is given by

$$\rho_o = \frac{p_o}{RT_o}$$

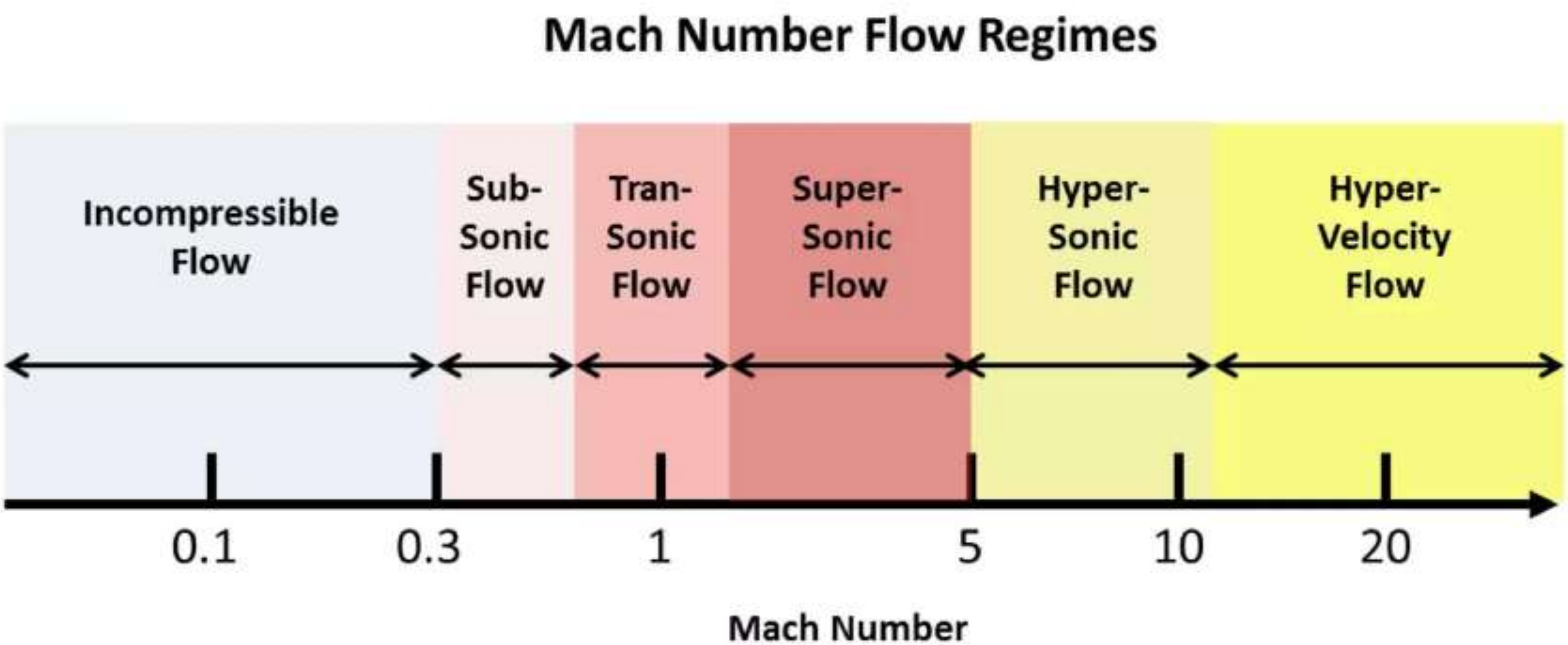
From adiabatic relation

$$\frac{T_o}{T} = \left(\frac{p_o}{p}\right)^{\frac{(\gamma-1)}{\gamma}} = \left(\frac{\rho_o}{\rho}\right)^{(\gamma-1)}$$

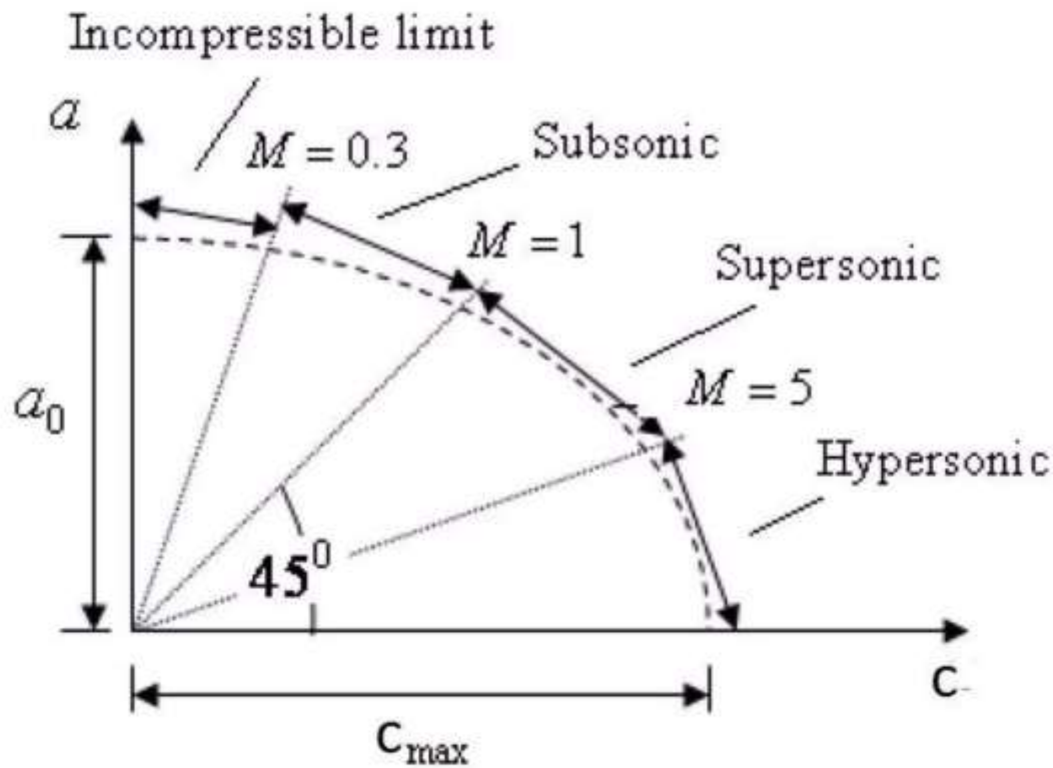
$$\left(\frac{\rho_0}{\rho}\right) = \left(\frac{T_0}{T}\right)^{\frac{1}{(\gamma-1)}} = \left[1 + \frac{(\gamma-1)}{2} M^2\right]^{\frac{1}{(\gamma-1)}} \quad .....(xv)$$

Dheenathayalan

Various regions of flow:



Various regions of flow:



Various regions of flow:

The adiabatic energy equation for a perfect gas is derived in terms of fluid velocity (c) and sound velocity (a).

Form adiabatic energy equation

$$h_o = h + \frac{c^2}{2} = constant \qquad .....(i)$$

We know that,

$$h = C_p T = \frac{\gamma}{(\gamma - 1)} RT = \frac{a^2}{(\gamma - 1)}$$

By substitution this in equation (i)

$$h_o = \frac{a^2}{(\gamma - 1)} + \frac{c^2}{2} = constant \qquad .....(ii)$$

At T=0; h=0; a=0 and $c=c_{max}$

Dheenathayalan

Various regions of flow:

Therefore equation (ii) becomes

$$h_o = \frac{c_{\max}^2}{2} \qquad \ldots\ldots(iii)$$

At c=0; a=a_o

Therefore, form equation (iii)

$$h_o = \frac{a_o^2}{(\gamma - 1)} = constant \qquad \ldots\ldots(iv)$$

$$\boxed{h_o = \frac{a^2}{(\gamma - 1)} + \frac{c^2}{2} = \frac{c_{\max}^2}{2} = \frac{a_o^2}{(\gamma - 1)} = constant} \qquad \ldots\ldots(v)$$

Various regions of flow:

M < 0.3 – Incompressible flow region

0.3 < M <0.8 – Subsonic flow region

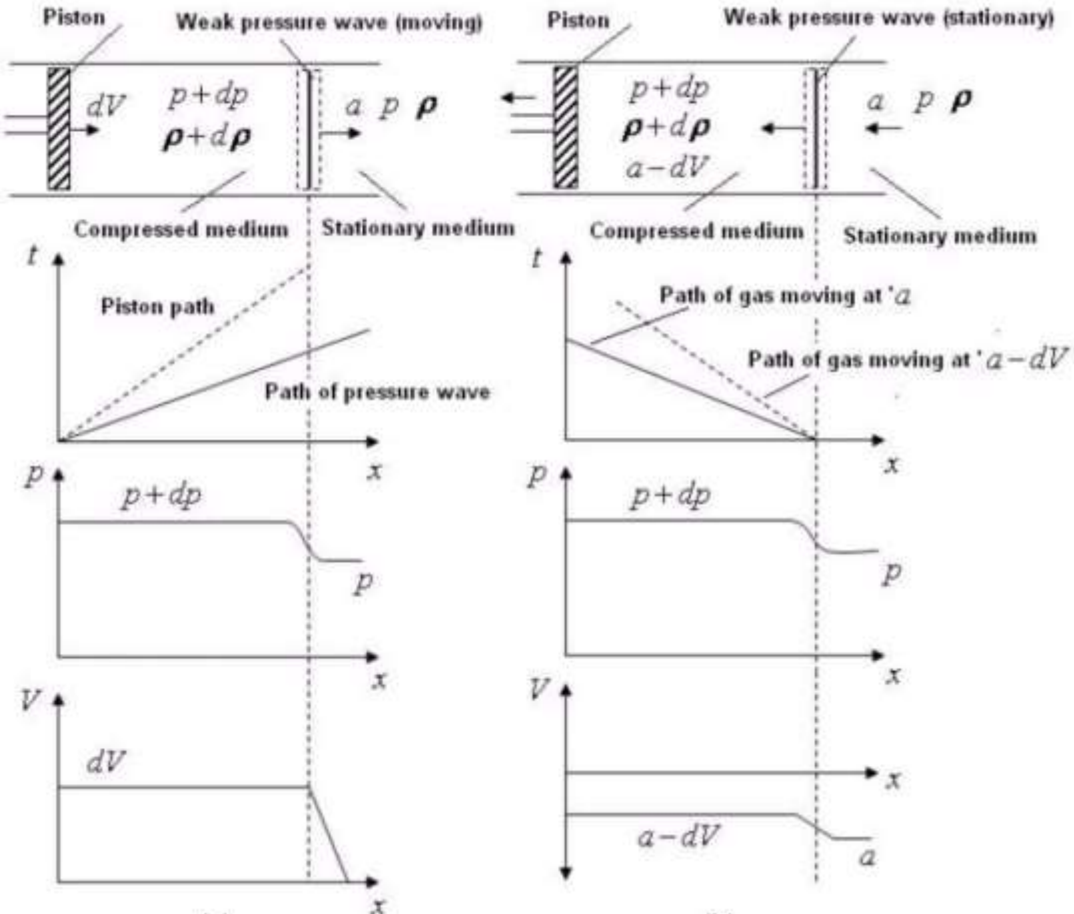
0.8 < M < 1.2 – Transonic flow region

1.2 < M < 5 – super sonic flow region

5 < M – Hypersonic flow region

Acoustic velocity (or) Sound velocity:

Wave front – a plane cross which pressure and density changes suddenly and there will be discontinuity in pressure, temperature and density.



(a) Moving Wave

(b) Stationary Wave

Acoustic velocity (or) Sound velocity:

- ✓ If small impulse is given to the piston the gas immediately adjacent to the piston will experience a slight rise in pressure (dp) or in other word it is compressed.

- ✓ The change in (dρ) takes place because the gas is compressible and therefore, there is lapse of time between the motion of the piston and the time this is observed at the far end of the tube.

- ✓ Thus it will take certain time to reach far end of the tube or in other words there is finite velocity of propagation which is acoustic velocity.

Acoustic velocity (or) Sound velocity:

- ✓ In this case the segment gas at pressure p on the right side moving with velocity 'a' toward left and thus its pressure is raised to (p+dp) and its velocity lowered to (a-dc).
- ✓ This because of the velocity of piston acts opposite to the movement of gas.

Before deriving following assumption made:

1. The fluid velocity is assumed to be acoustic velocity.
2. There is no heat transfer in the pipe and the flow is through a constant area pipe.
3. The change across an infinitesimal pressure wave can be assumed as reversible adiabatic (or) isentropic.

SOUND - PRODUCTION AND PROPAGATION OF SOUND

Transverse and Longitudinal Waves

Dheenathayalan

Acoustic velocity (or) Sound velocity:

$$pA - (p + dp)A = \dot{m}[(a - dc) - a]$$

Pressure force — Impulse force

$$[\dot{m} = \rho Aa]$$

$$A[p - p - dp] = \rho Aa[a - dc - a] \qquad [\therefore A = constant]$$

$$\therefore -dp = -\rho adc$$

$$\therefore dp = \rho adc \qquad \ldots\ldots(i)$$

From continuity equation for the two sides of the wave

$$\dot{m} = \rho Aa = (\rho + d\rho)A(a - dc)$$

$$\rho a = \rho a + ad\rho - \rho dc - d\rho dc \qquad \ldots\ldots(ii)$$

Acoustic velocity (or) Sound velocity:

The product of dρ dc is very small, hence it is ignored. The eqn (ii) becomes

$$a d\rho = \rho dc$$

Substituting this in equation (i), we get

$$dp = a^2 d\rho$$

$$\therefore a = \left(\sqrt{\frac{dp}{d\rho}}\right) \qquad \ldots\ldots(iii)$$

For an isentropic flow,

Dheenathayalan

Acoustic velocity (or) Sound velocity:

$$pv^{\gamma} = C \ \ (or) \ \frac{p}{\rho^{\gamma}} = constant$$

$$p\rho^{-\gamma} = constant$$

Differentiating above equation

$$p[-\gamma\rho^{-\gamma-1}d\rho] + \rho^{-\gamma}(dp) = 0$$

$$-p\gamma\rho^{-\gamma} \times \rho^{-1}d\rho + \rho^{-\gamma}dp = 0$$

$$dp = \frac{\gamma p}{\rho} \times d\rho$$

$$\frac{dp}{d\rho} = \gamma\frac{p}{\rho}$$

$$\frac{dp}{d\rho} = \gamma RT$$

$$[p = \rho RT]$$

$$\left[\frac{p}{\rho} = RT\right]$$

Acoustic velocity (or) Sound velocity:

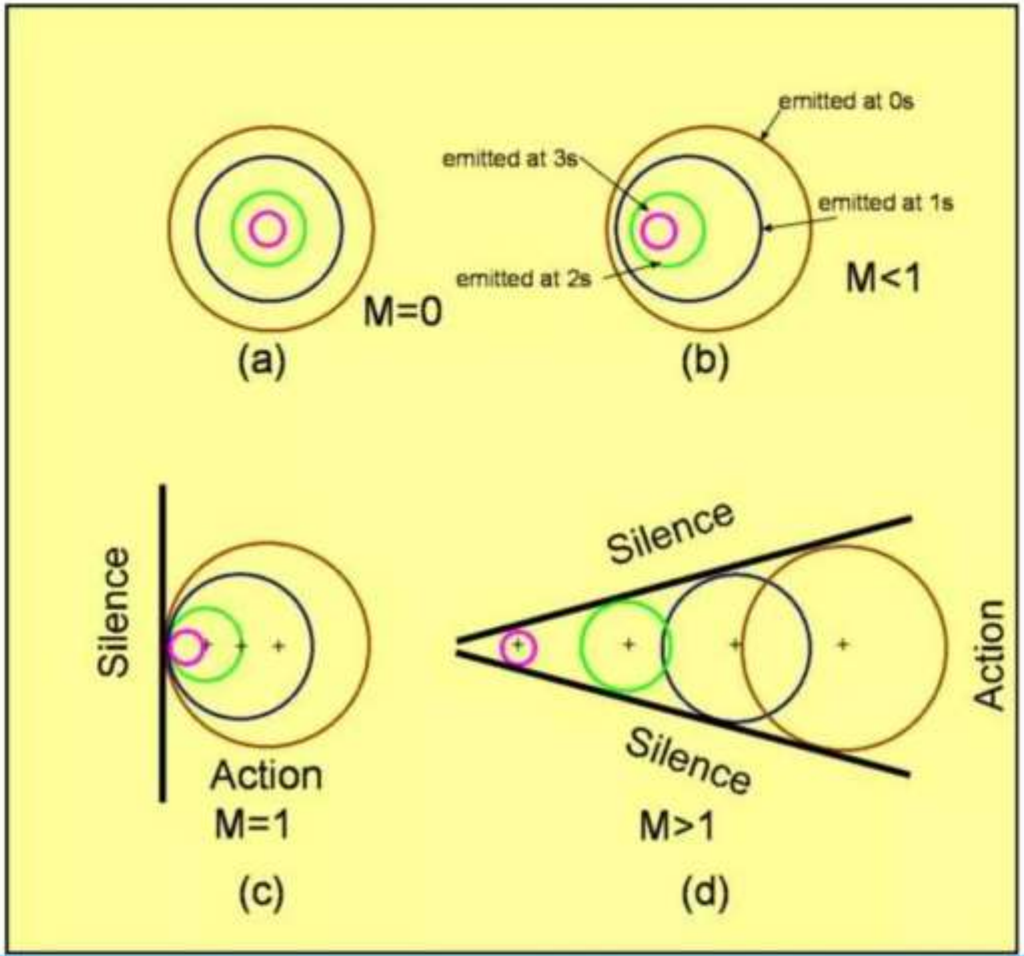
Substitution this in equation (iii)

$$a = \sqrt{\frac{dp}{d\rho}} = \sqrt{\gamma RT} \qquad \ldots\ldots(iv)$$

The velocity of sound in normal ambient temperature is about 340 m/s.


Mach-like gravity-capillary wakes
Frédéric Moisy, Marc Rabaud
Université Paris-Sud, CNRS, Laboratoire FAST
Physical Review E 90, 023009 (2014)

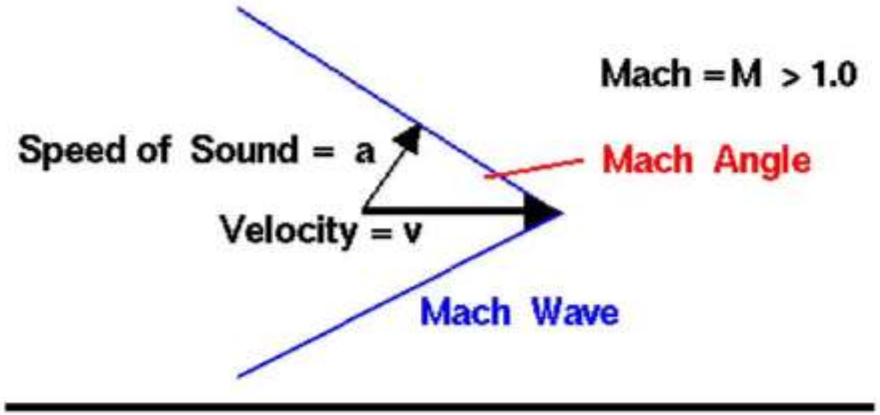
Mach angle and Mach cone



a) Incompressible flow
b) Subsonic flow
c) Sonic flow
d) Supersonic flow

Dheenathayalan

Mach angle



$$\sin\alpha = \frac{a}{v}$$

$$\sin\alpha = \frac{1}{M}$$

Mach angle $\alpha = \sin^{-1}\frac{1}{M}$

Maximum velocity of fluid, C_{max}

From adiabatic energy equation

$$h_o = h + \frac{c^2}{2}$$

It has two components one is enthalpy (h) and the another is kinetic energy $\frac{c^2}{2}$. When the static enthalpy is zero (or) when the entire energy is mad up of kinetic energy only the above equation becomes

$$h = 0 \quad \text{and} \quad c = c_{\max}$$

$$\frac{C_{\max}}{a_o} = \sqrt{\frac{2}{(\gamma - 1)}} \qquad h_0 = \frac{c_{\max}^2}{2}$$

Maximum velocity of fluid, C_{max}

$$h_0 = \frac{c_{\max}^2}{2}$$

$$\left[h_o = \frac{a^2}{(\gamma - 1)} + \frac{c^2}{2} = \frac{c_{\max}^2}{2} = \frac{a_o^2}{(\gamma - 1)} \right]$$

$$c_{\max} = \sqrt{2h_0}$$

$$c_{\max} = \sqrt{2c_p T_0} = \sqrt{2 \times \frac{\gamma}{(\gamma - 1)} R T_0}$$

$$c_{\max} = \left(\sqrt{\frac{2}{(\gamma - 1)}} \right) a_0$$

$$\frac{C_{\max}}{a_o} = \sqrt{\frac{2}{(\gamma - 1)}}$$

Dheenathayalan

Crocco Number, $[C_r]$

Crocco number is a non-dimensional fluid velocity which is defined as the ratio of fluid velocity to its maximum fluid velocity.

$$c_r = \frac{c}{c_{\max}}$$

$$M = \sqrt{\frac{2c_r^2}{(1 - c_r^2)(\gamma - 1)}}$$

$$\frac{T_o}{T} = \frac{1}{1 - c_r^2}$$

Practical matters:

- This course:

- Lectures on Wednesday, HG01.028; 15.30-17.30;
- Assignment course (werkcollege): *when and where* to be determined;

- Lecture Notes and PowerPoint slides on:
  www.astro.ru.nl/~achterb/Gasdynamica_2013

Overview

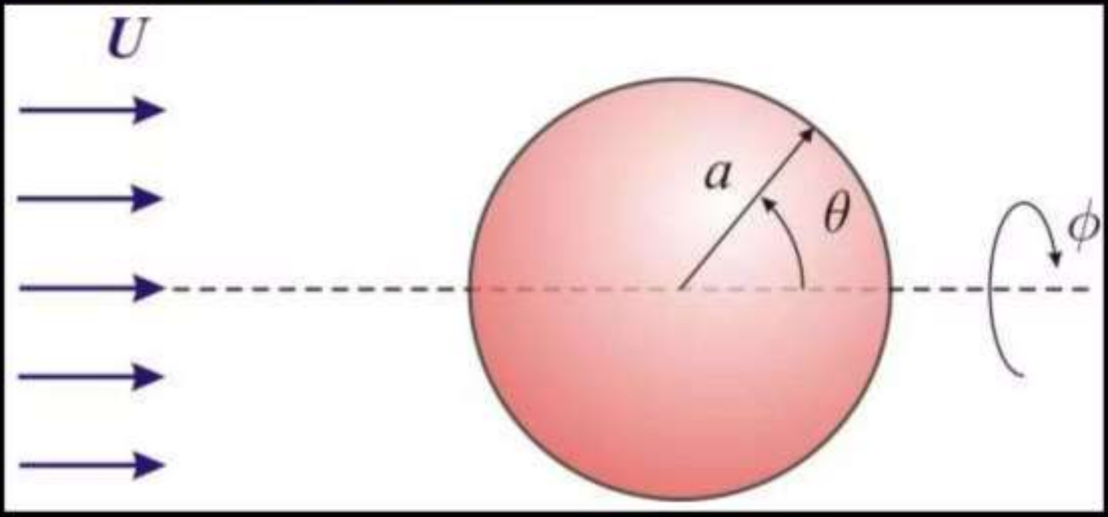
What will we treat during this course?

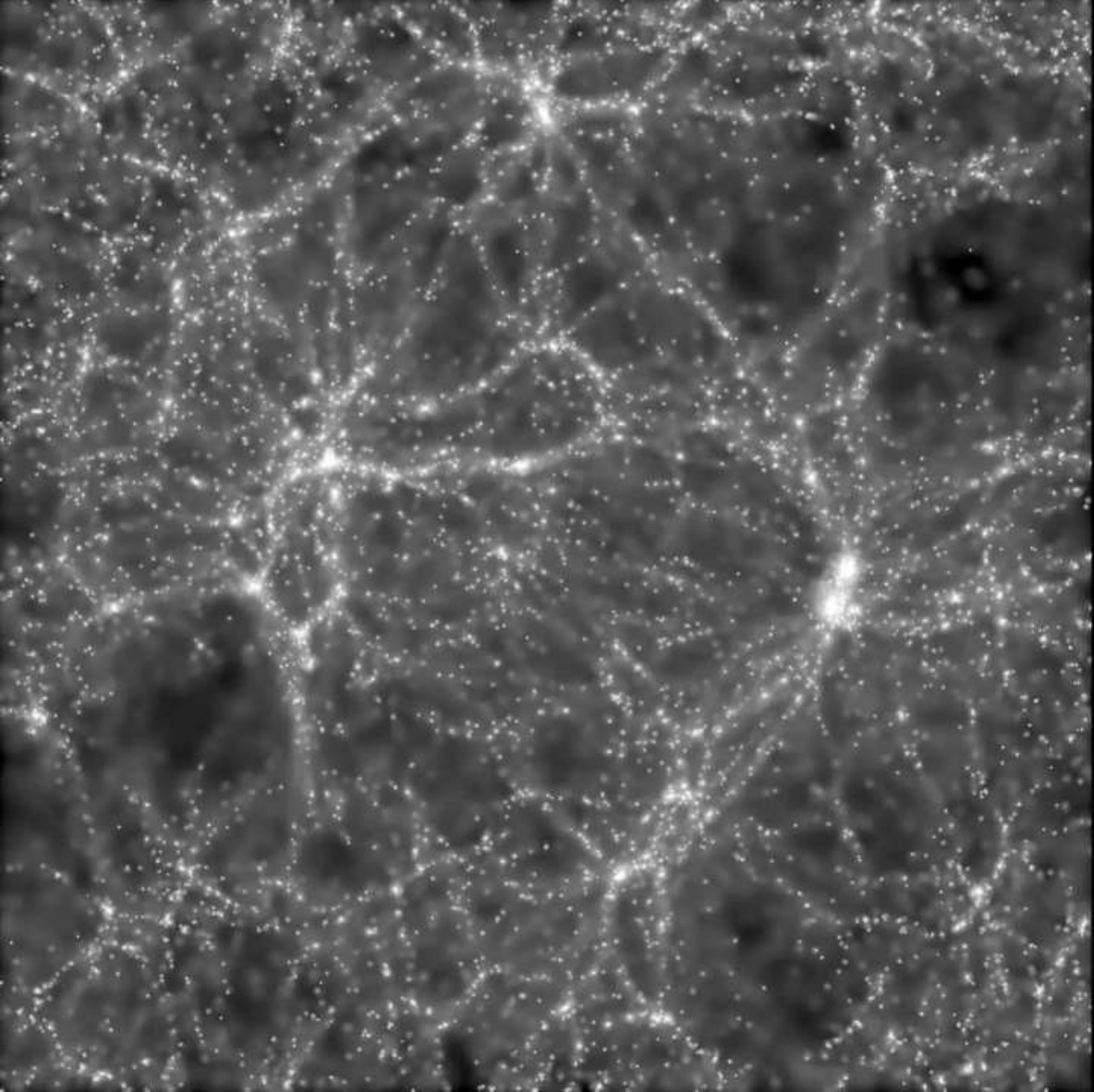
- Basic equations of gas dynamics
- Equation of motion
- Mass conservation
- Equation of state

- Fundamental processes in a gas
- Steady Flows
- Self-gravitating gas
- Wave phenomena
- Shocks and Explosions
- Instabilities: Jeans' Instability

Applications

- Isothermal sphere & Globular Clusters
- Special flows and drag forces
- Solar & Stellar Winds
- Sound waves and surface waves on water
- Shocks
- Point Explosions, Blast waves & Supernova Remnants



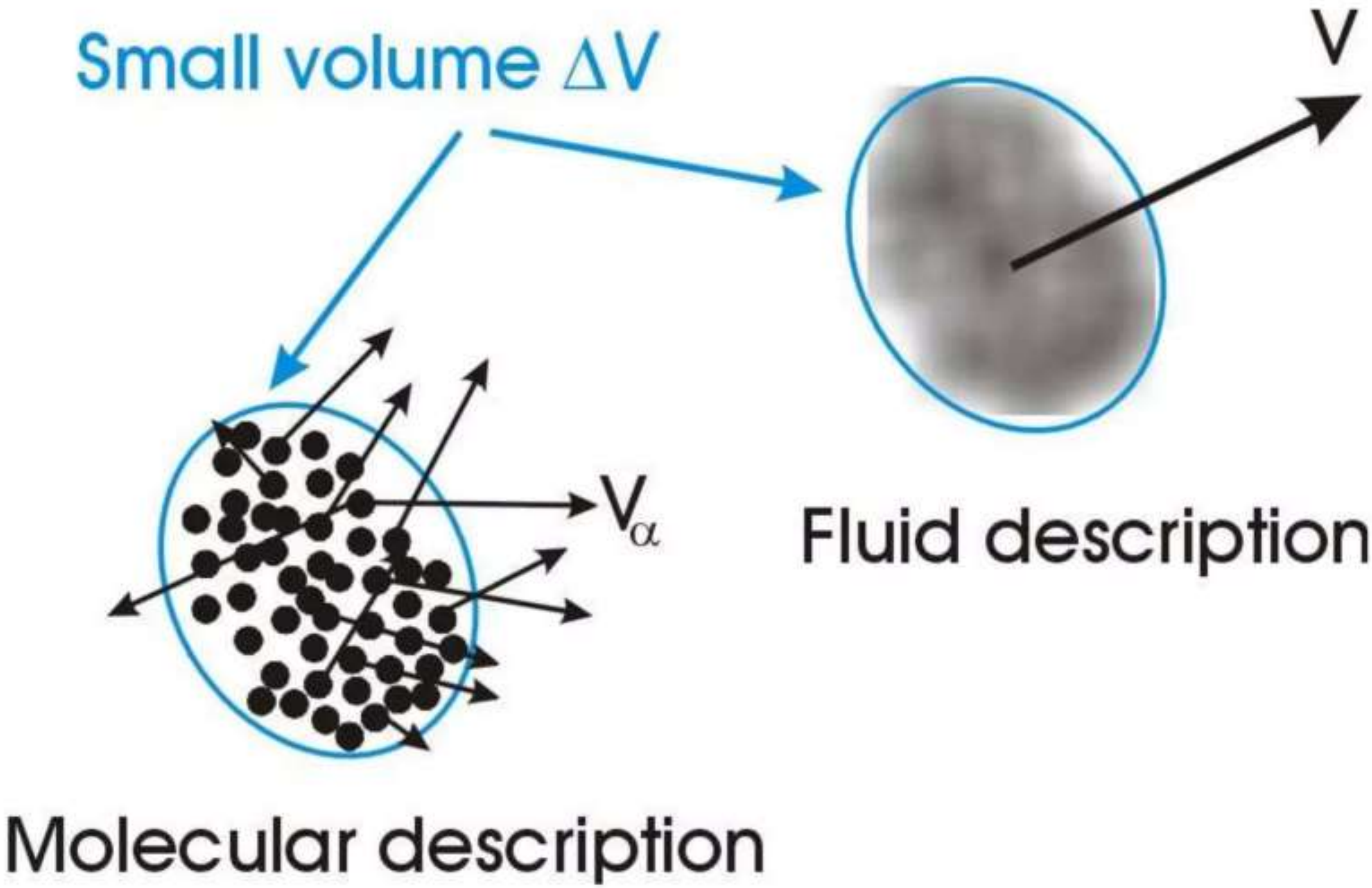


LARGE
SCALE
STRUCTURE

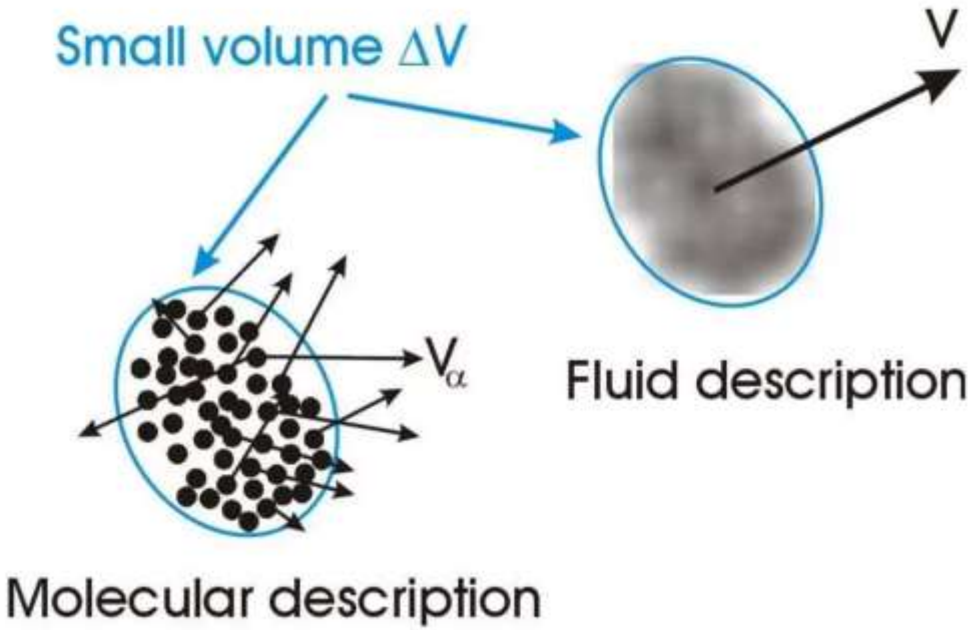
Classical Mechanics vs. Fluid Mechanics

Single-particle (classical) Mechanics	Fluid Mechanics
Deals with single particles with a fixed mass	Deals with a continuum with a variable mass-density
Calculates a single particle trajectory	Calculates a collection of flow lines (flow field) in space
Uses a position *vector* and velocity *vector*	Uses a *fields* : Mass density, velocity field....
Deals only with externally applied forces (e.g. gravity, friction etc)	Deals with internal AND external forces
Is formally linear (so: there is a *superposition principle* for solutions)	Is intrinsically non-linear No superposition principle in general!

Basic Definitions



Mass, mass-density and velocity



Mass density ρ:

$$\rho(\boldsymbol{x}\,,\,t) = \lim_{\Delta\mathcal{V}\to 0} \frac{\Delta m}{\Delta\mathcal{V}}$$

Mass Δm in volume ΔV

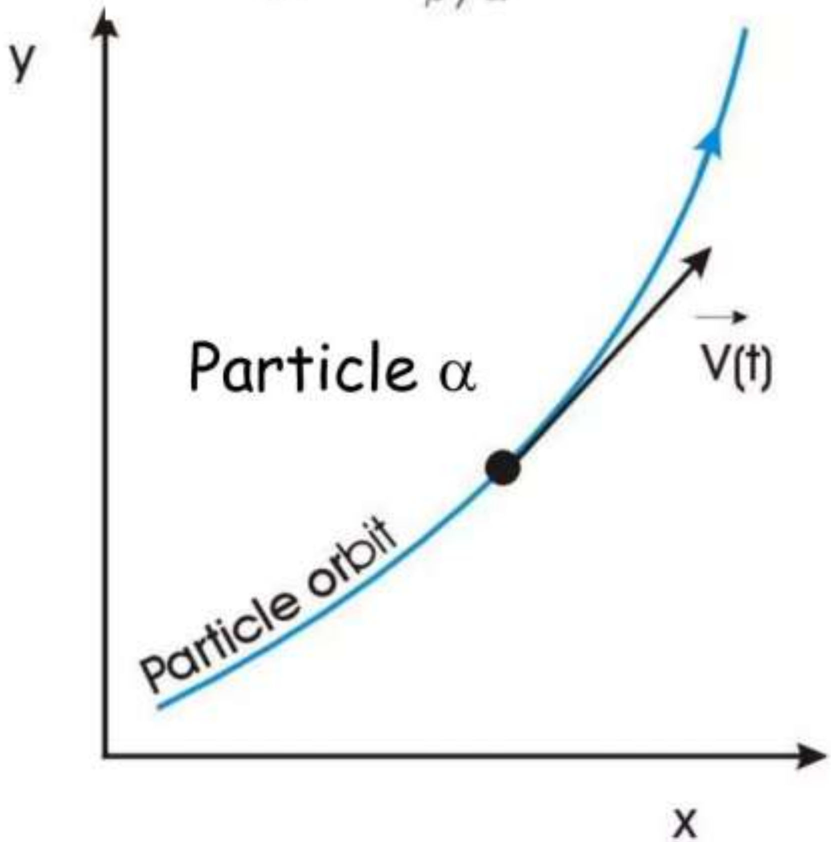
$$\Delta m = \sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha$$

Mean velocity $\boldsymbol{V}(\boldsymbol{x}\,,\,t)$ is defined as:

$$\boldsymbol{V} = \frac{\sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha \boldsymbol{V}_\alpha}{\Delta m}$$

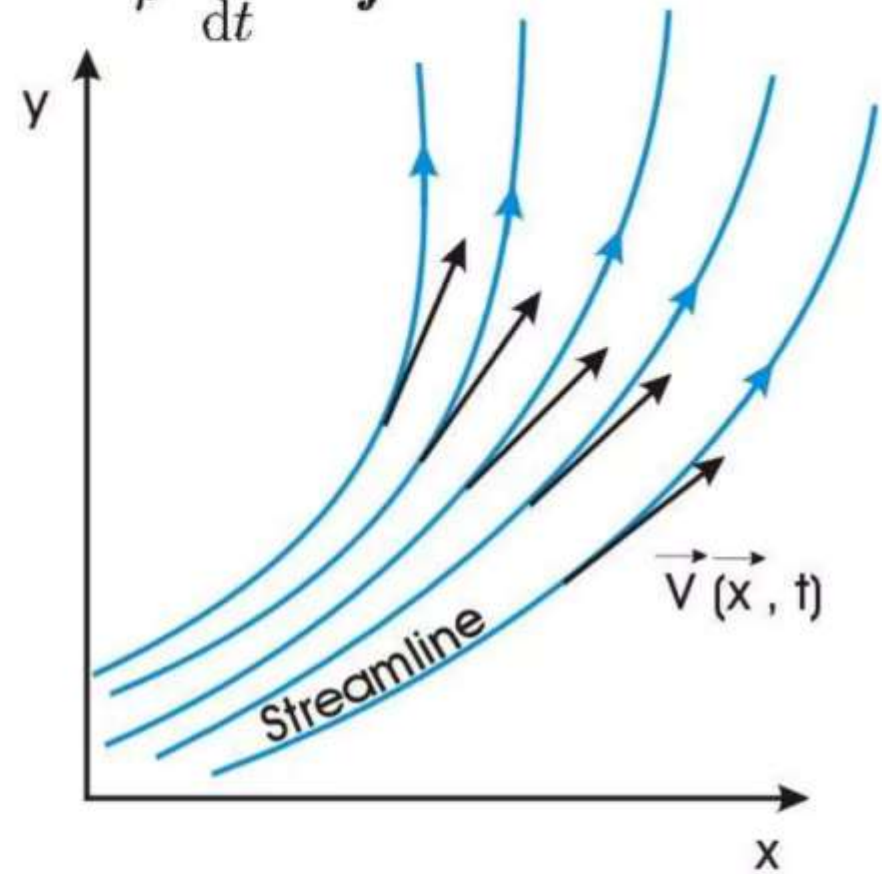
Equation of Motion: from Newton to Navier-Stokes/Euler

$$m_\alpha \frac{\mathrm{d}\boldsymbol{V}_\alpha}{\mathrm{d}t} = \sum_{\beta \neq \alpha} \boldsymbol{F}_{\alpha\beta}$$



Single-particle dynamics

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$



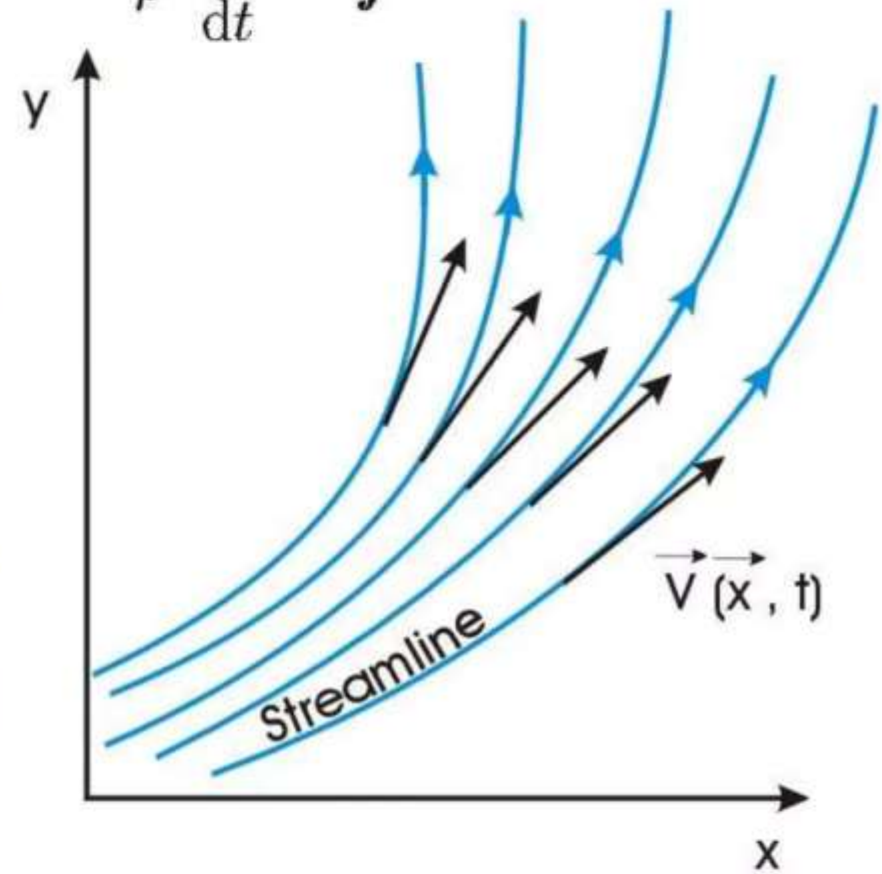
Fluid dynamics

Equation of Motion: from Newton to Navier-Stokes/Euler

You have to work with a velocity <u>field</u> that depends on position *and* time!

$$\boldsymbol{V} = (V_\mathrm{x}\,,\, V_\mathrm{y}\,,\, V_\mathrm{z}) = \boldsymbol{V}(\boldsymbol{x}\,,\, t)$$

$$\rho\,\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$



Fluid dynamics

Derivatives, derivatives…

Eulerian change: $$\delta Q = Q(\boldsymbol{x}\,,\, t+\Delta t) - Q(\boldsymbol{x}\,,\, t) \approx \frac{\partial Q}{\partial t}\,\Delta t$$

Derivatives, derivatives…

Eulerian change:
evaluated at a
<u>fixed position</u>

$$\delta Q = Q(\boldsymbol{x}\,,\, t + \Delta t) - Q(\boldsymbol{x}\,,\, t) \approx \frac{\partial Q}{\partial t}\,\Delta t$$

Lagrangian change:
evaluated at a
<u>shifting position</u>

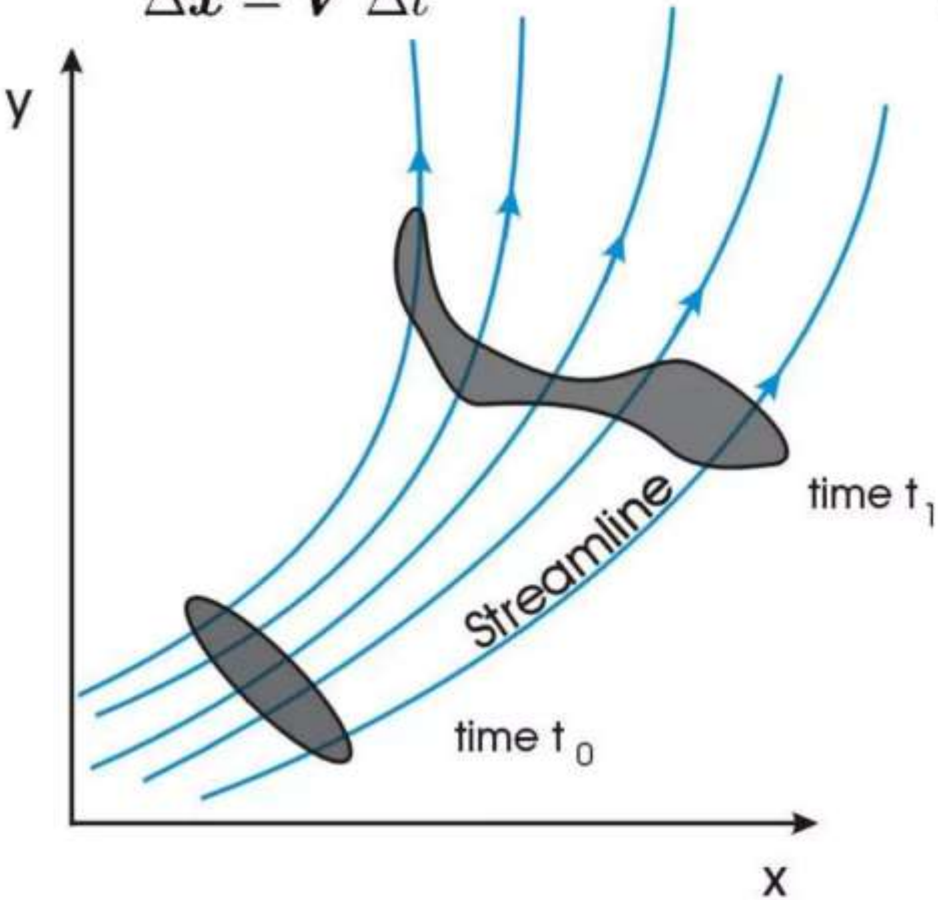
$$\Delta Q = Q(\boldsymbol{x} + \Delta \boldsymbol{x}\,,\, t + \Delta t) - Q(\boldsymbol{x}\,,\, t) \approx \frac{\mathrm{d}Q}{\mathrm{d}t}\,\Delta t$$

Shift along
streamline:

$$\boxed{\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t}$$

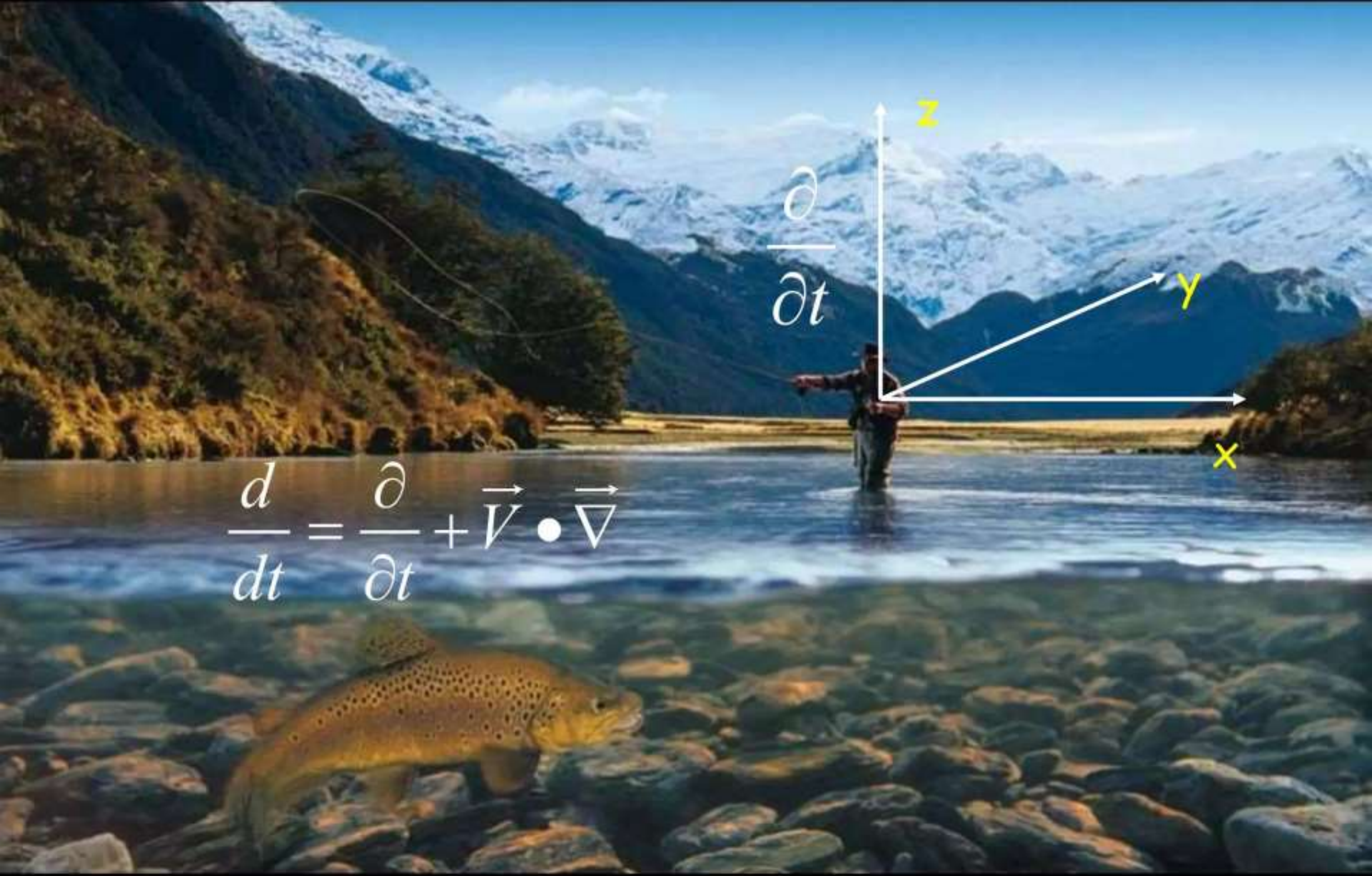
Comoving derivative d/dt

$$\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t$$



$$\begin{aligned}\Delta Q &= Q(t+\Delta t\,,\,\boldsymbol{x}+\Delta\boldsymbol{x}) - Q(t\,,\,\boldsymbol{x}) \\ &\approx \frac{\partial Q}{\partial t}\Delta t + (\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla})Q \\ &= \left[\frac{\partial Q}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})Q\right]\Delta t \\ &\equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)\Delta t\,.\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})}$$


$\frac{\partial}{\partial t}$
z
y
x
$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \vec{\nabla}$

Notation: working with the gradient operator

Gradient operator is a 'machine' that converts a scalar into a vector:

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

For scalar $Q(\boldsymbol{x}\ , t)$:

$$\boldsymbol{\nabla} Q = \frac{\partial Q}{\partial x}\,\hat{\mathbf{e}}_x + \frac{\partial Q}{\partial y}\,\hat{\mathbf{e}}_y + \frac{\partial Q}{\partial z}\,\hat{\mathbf{e}}_z$$

Related operators: turn scalar into scalar, vector into vector….

$$\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla} \equiv \Delta x\,\frac{\partial}{\partial x} + \Delta y\,\frac{\partial}{\partial y} + \Delta z\,\frac{\partial}{\partial z}$$

$$\boldsymbol{V}\cdot\boldsymbol{\nabla} \equiv V_{\mathrm{x}}\frac{\partial}{\partial x} + V_{\mathrm{y}}\frac{\partial}{\partial y} + V_{\mathrm{z}}\frac{\partial}{\partial z}$$

GRADIENT OPERATOR AND VECTOR ANALYSIS (See Appendix A)

scalar into vector: $\boldsymbol{g} = -\tilde{N}\,\Phi$

vector into scalar: $\tilde{N} \bullet \boldsymbol{g} = -4\pi G\rho$

vector into vector: $\tilde{N} \times \boldsymbol{B} = \dfrac{4\pi}{c}\boldsymbol{J}$

tensor into vector: $\tilde{N} \bullet \mathbf{T} = -\boldsymbol{f}$

Useful relations: $\tilde{N} \bullet \left(\tilde{N} \times \boldsymbol{B}\right) = 0\,, \quad \tilde{N} \times \tilde{N}\,\Phi = 0\,, \tilde{N} \bullet \left(\tilde{N}\,\Phi\right) = \nabla^2\Phi$

Program for uncovering the basic equations:

1. Define the fluid acceleration and formulate the equation of motion by analogy with single particle dynamics;

2. Identify the forces, such as pressure force;

3. Find equations that describe the response of the other fluid properties (such as: density □, pressure P, temperature T) to the flow.

Equation of motion for a fluid:

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}$$

Equation of motion for a fluid:

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}$$

The acceleration of a fluid element is defined as:

$$\boldsymbol{a} \equiv \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \bullet \tilde{N})\boldsymbol{V}$$

Equation of motion for a fluid:

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}$$

This equation states:

mass density × acceleration = force density

note: GENERALLY THERE IS NO
FIXED MASS IN FLUID MECHANICS!

Equation of motion for a fluid:

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}$$

Non-linear term!

Makes it much more difficult
To find 'simple' solutions.

Prize you pay for working with
a velocity-field

$$\boldsymbol{V} = (V_{\rm x} \, , \, V_{\rm y} \, , \, V_{\rm z}) = \boldsymbol{V}(\boldsymbol{x} \, , \, t)$$

Equation of motion for a fluid:

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}\right] = \boldsymbol{f}$$

Non-linear term!

Makes it much more difficult
To find 'simple' solutions.

Prize you pay for working with
a velocity-field

$$\boldsymbol{V} = (V_{\mathrm{x}} \, , \, V_{\mathrm{y}} \, , \, V_{\mathrm{z}}) = \boldsymbol{V}(\boldsymbol{x} \, , \, t)$$

Force-density

This force densitycan be:
- internal:
- pressure force
- viscosity (friction)
- self-gravity

- external
- For instance: external gravitational force

Pressure force and thermal motions

Split velocities into the
<u>average velocity</u>

$V(\boldsymbol{x}, t)$,

and an
isotropically distributed
deviation from average,
the
<u>random velocity:</u>

$\sigma(\boldsymbol{x}, t)$

Individual particle:

$$\boldsymbol{v}_\alpha = \boldsymbol{V}(\boldsymbol{x}\,,\,t) + \boldsymbol{\sigma}_\alpha(\boldsymbol{x}\,,\,t)\,.$$

Average properties of random velocity $\boldsymbol{\sigma}$:

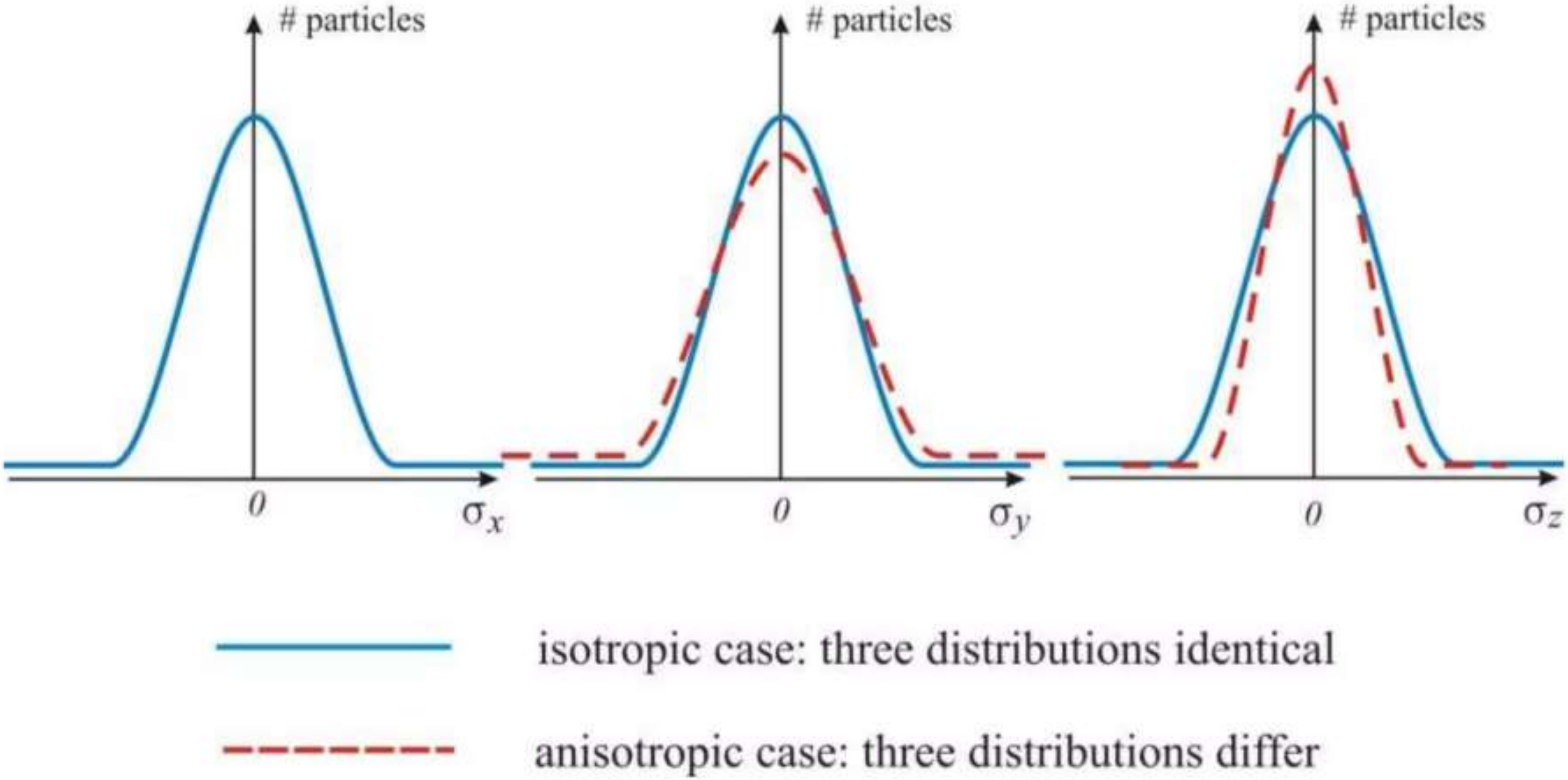
$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{v}} - \boldsymbol{V} = \boldsymbol{0}\,;$$

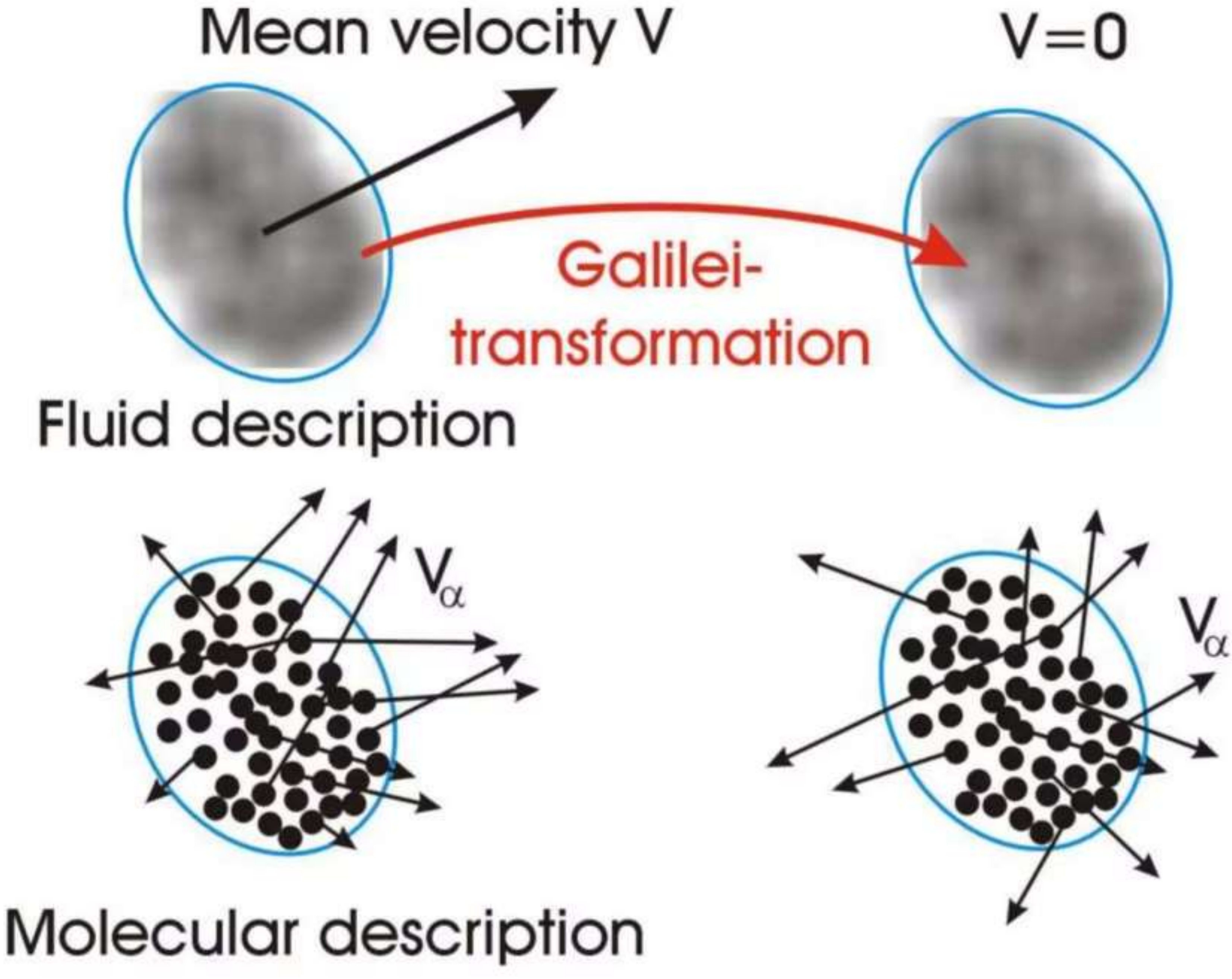
$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = \tfrac{1}{3}\overline{\sigma^2}\,,$$

and

$$\overline{\sigma_x \sigma_y} = \overline{\sigma_x \sigma_z} = \overline{\sigma_y \sigma_z} = \cdots = 0\,.$$

DISTRIBUTION OF RANDOM VELOCITIES ALONG THE THREE COORDINATE AXES




Mean velocity V
V=0
Galilei-
transformation
Fluid description
V_α
V_α
Molecular description

Acceleration of particle α

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{v}_\alpha}{\mathrm{d}t} &= \frac{\partial \boldsymbol{v}_\alpha}{\partial t} + (\boldsymbol{v}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{v}_\alpha \\
&= \frac{\partial(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)}{\partial t} + ((\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \cdot \boldsymbol{\nabla})(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \\
&= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\frac{\partial \boldsymbol{\sigma}_\alpha}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}_\alpha}_{\text{linear in } \boldsymbol{\sigma}}} + \boxed{\underbrace{(\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}_\alpha}_{\text{quadratic in } \boldsymbol{\sigma}}}
\end{aligned}$$

Acceleration of particle α (II)

Effect of average over many particles in small volume:

$$\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}}$$

$$= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \underbrace{\left(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\right)\overline{\boldsymbol{\sigma}}}_{\text{vanishes: }\overline{\boldsymbol{\sigma}}=0!} + \underbrace{\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\boldsymbol{\sigma}}}_{\text{remains: quadratic in }\boldsymbol{\sigma}}$$

Average equation of motion:

$$\rho\,\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\boldsymbol{f}}$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = \underbrace{\overline{\boldsymbol{f}}}_{\text{mean ext. force}} \quad -\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\boldsymbol{\sigma}}$$

For isotropic fluid: $\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P$

Some tensor algebra:

Vector

$$\boldsymbol{A} \equiv A_i\, \boldsymbol{e}_i = A_x\, \boldsymbol{e}_1 + A_y \boldsymbol{e}_2 + A_z\, \boldsymbol{e}_3 = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Three notations for the same animal!

Some tensor algebra:
the divergence of a vector in cartesian (x, y, z) coordinates

Vector

$$\boldsymbol{A} \equiv A_i\, \boldsymbol{e}_i = A_x\, \boldsymbol{e}_1 + A_y \boldsymbol{e}_2 + A_z\, \boldsymbol{e}_3 = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

↓

Scalar

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Rank 2 Tensor

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j == \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Rank 2 Tensor and Tensor Divergence

Rank 2 tensor $\boldsymbol{T}$

↓

Vector

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j == \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = \left(\frac{\partial T_{ij}}{\partial x_i}\right)\,\boldsymbol{e}_j = \begin{pmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{pmatrix}$$

Special case:
Dyadic Tensor = Direct Product of two Vectors

$$\boldsymbol{A} \otimes \boldsymbol{B} \equiv A_i B_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \, \boldsymbol{B} + (\boldsymbol{A} \cdot \boldsymbol{\nabla}) \, \boldsymbol{B}$$

This is the product rule for differentiation!

Application: Pressure Force (I)

Tensor divergence:

$$(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\,\boldsymbol{\sigma}$$

Isotropy of the random velocities:

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})$$

Second term = scalar x vector!

This <u>must</u> vanish upon averaging!!

Application: Pressure Force (II)

Isotropy of the random velocities

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\left(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}\right)$$

$$\overline{\sigma_i\sigma_j} = \tfrac{1}{3}\overline{\sigma^2}\,\delta_{ij} = \begin{cases} \tfrac{1}{3}\overline{\sigma^2} & \text{when } i=j \\ 0 & \text{when } i\neq j \end{cases}$$

$$\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} = \rho\begin{pmatrix} \tfrac{1}{3}\overline{\sigma^2} & 0 & 0 \\ 0 & \tfrac{1}{3}\overline{\sigma^2} & 0 \\ 0 & 0 & \tfrac{1}{3}\overline{\sigma^2} \end{pmatrix} = \frac{\rho\overline{\sigma^2}}{3}\,\boldsymbol{I}$$

Diagonal Pressure Tensor

Pressure force, conclusion:

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\left(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}\right) = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P$$

Equation of motion for frictionless ('ideal') fluid:

$$\rho\left(\frac{\partial\boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = -\boldsymbol{\nabla}P + \text{other (external) forces}$$

$$P(\boldsymbol{x}\,,\,t) \equiv \tfrac{1}{3}\rho\,\overline{\sigma^2}$$

Summary:

- We know how to interpret the time-derivative $\mathrm{d}/\mathrm{d}t$;
- We know what the equation of motion looks like;
- We know where the pressure force comes from (thermal motions), and how it looks: $\boldsymbol{f} = - \Box P$.
- **We <u>still</u> need:**

  - A way to link the pressure to density and temperature: $P = P(\Box, T);$
  - A way to calculate how the density □ of the fluid changes.

What did we learn last time around?

-Equation of motion;

-Relation between pressure
and thermal velocity dispersion;

-Form of the pressure force

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right)=-\boldsymbol{\nabla}P+\text{other (external) forces}$$

$$P(\boldsymbol{x}\,,\,t)\equiv\tfrac{1}{3}\rho\,\overline{\sigma^2}$$

A little thermodynamics: ideal gas law

Each degree of freedom carries an energy $\frac{1}{2} k_{\mathrm{b}} T$

Point particles with mass m:

$$\tfrac{1}{2} m \left\langle \sigma_x^2 \right\rangle = \tfrac{1}{2} m \left\langle \sigma_y^2 \right\rangle = \tfrac{1}{2} m \left\langle \sigma_z^2 \right\rangle = \tfrac{1}{2} k_{\mathrm{b}} T$$

$$\Leftrightarrow$$

$$\frac{\sigma^2}{3} = \frac{k_{\mathrm{b}} T}{m} \quad \Leftrightarrow \quad \boxed{P = \rho \frac{\sigma^2}{3} = \rho \left(\frac{k_{\mathrm{b}} T}{m} \right) = n k_{\mathrm{b}} T}$$

Alternative way to write this:

$$P = \frac{\rho \mathrm{R}\, T}{\mu}$$

$$\mathrm{R} = \frac{k_\mathrm{b}}{m_\mathrm{H}} = \text{ universal gas constant;}$$

$$\mu = \frac{m}{m_\mathrm{H}} = \text{ mass in units of mass hydrogen atom.}$$

Some more thermodynamics (see *Lecture Notes*)

Adiabatic change: no energy is irreversibly lost from the system, or gained by the system

$$\mathrm{d}U + P\mathrm{dV} = 0$$

Some more thermodynamics (see *Lecture Notes*)

Adiabatic change: no energy is irreversibly lost from the system, or gained by the system

$$\mathrm{d}U + P\mathrm{dV} = 0$$

Change in internal energy U

Work done by pressure forces in volume change $\mathrm{d}\varsigma$

Gas of structure-less point particles

Thermal energy density:

$$W_{th} = n \underbrace{\left(\tfrac{1}{2} m\sigma^2\right)}_{\text{kinetic energy of thermal motion}} = \tfrac{3}{2} n k_b T = \tfrac{3}{2} \frac{\rho R\, T}{\mu}$$

Pressure:

$$P = \frac{\rho R\, T}{\mu} = \tfrac{2}{3} W_{th}$$

Thermal equilibrium:

$$P = \frac{\rho \mathcal{R} T}{\mu} \;\; , \;\; U = \tfrac{3}{2}\,\frac{\rho \mathcal{R} T\, \mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P\,\mathrm{d}\mathcal{V} = 0 \longrightarrow \mathrm{d}\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right)\mathrm{d}\mathcal{V} = 0$$

Thermal equilibrium:

$$P = \frac{\rho \mathcal{R} T}{\mu} \;, \;\; U = \tfrac{3}{2} \frac{\rho \mathcal{R} T \, \mathcal{V}}{\mu}$$

Adiabatic change:

$$\mathrm{d}U + P \, \mathrm{d}\mathcal{V} = 0 \longrightarrow \mathrm{d}\left(\frac{3\rho \mathcal{R} T \mathcal{V}}{2\mu}\right) + \left(\frac{\rho \mathcal{R} T}{\mu}\right) \mathrm{d}\mathcal{V} = 0$$

Product rule for 'd'-operator:

$$\mathrm{d}(f\, g) = (\mathrm{d}f)\, g + f\, (\mathrm{d}g) \longrightarrow \tfrac{5}{3}\, P \, \mathrm{d}\mathcal{V} + \mathcal{V}\, \mathrm{d}P = 0 \, .$$

(just like differentiation!)

$$\frac{\mathrm{d}P}{P} + \tfrac{5}{3} \frac{\mathrm{d}\mathcal{V}}{\mathcal{V}} = \mathrm{d} \log \left(P \, \mathcal{V}^{5/3} \right) = 0$$

Adiabatic Gas Law: a polytropic relation

Adiabatic pressure change:

$$P \times \mathcal{V}^{5/3} = \text{constant}$$

For <u>small</u> volume:
mass conservation!

$$M = \rho\, \mathcal{V} = \text{constant}$$

$$\boxed{P\, \rho^{-5/3} = \text{constant}}$$

General case for adiabatic changes:

Polytropic gas law: $P = K\rho^{\gamma}$

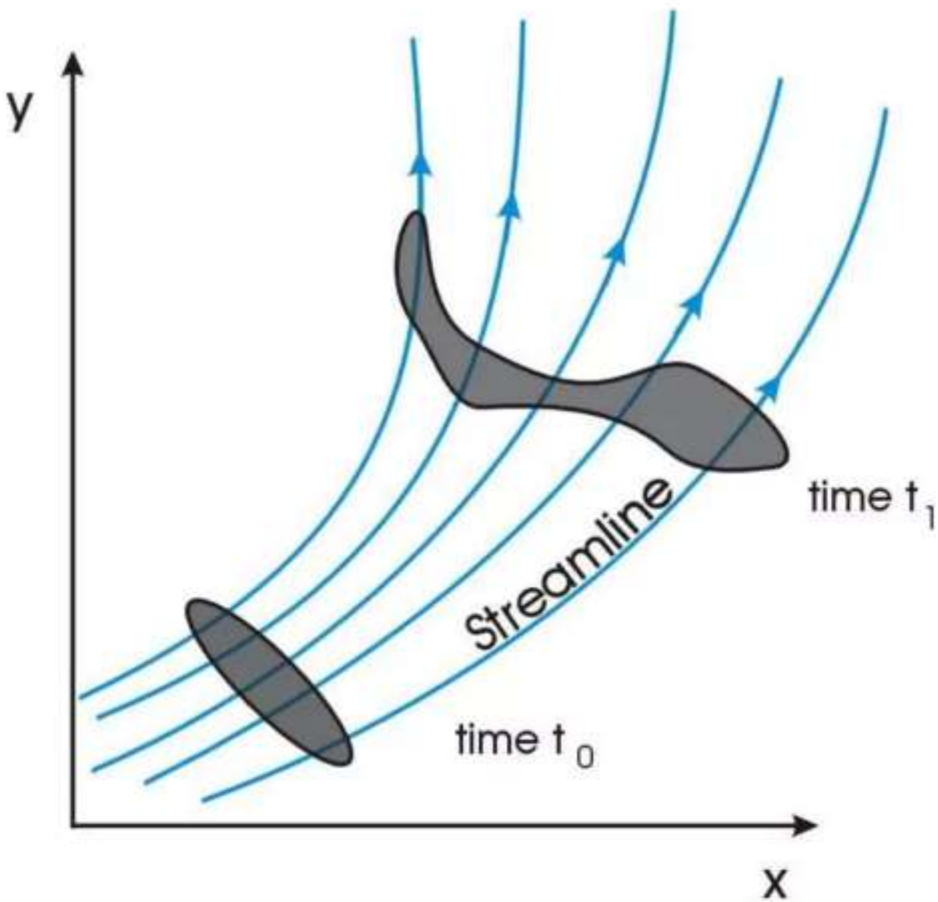
Ideal gas law: $P = \dfrac{\rho R\ T}{\mu}$

$$\left.\begin{aligned} P &= K\rho^{\gamma} \\ P &= \frac{\rho R\ T}{\mu} \end{aligned}\right\} \Rightarrow T = K'\rho^{\gamma-1}$$

Thermal energy density: $W_{\rm th} = \dfrac{P}{\gamma - 1} = \dfrac{\rho R\ T}{(\gamma-1)\mu}$

Polytropic index mono-atomic gas: $\gamma = \dfrac{5}{3}$, γ=1: ISOTHERMAL

Mass conservation and the volume-change law



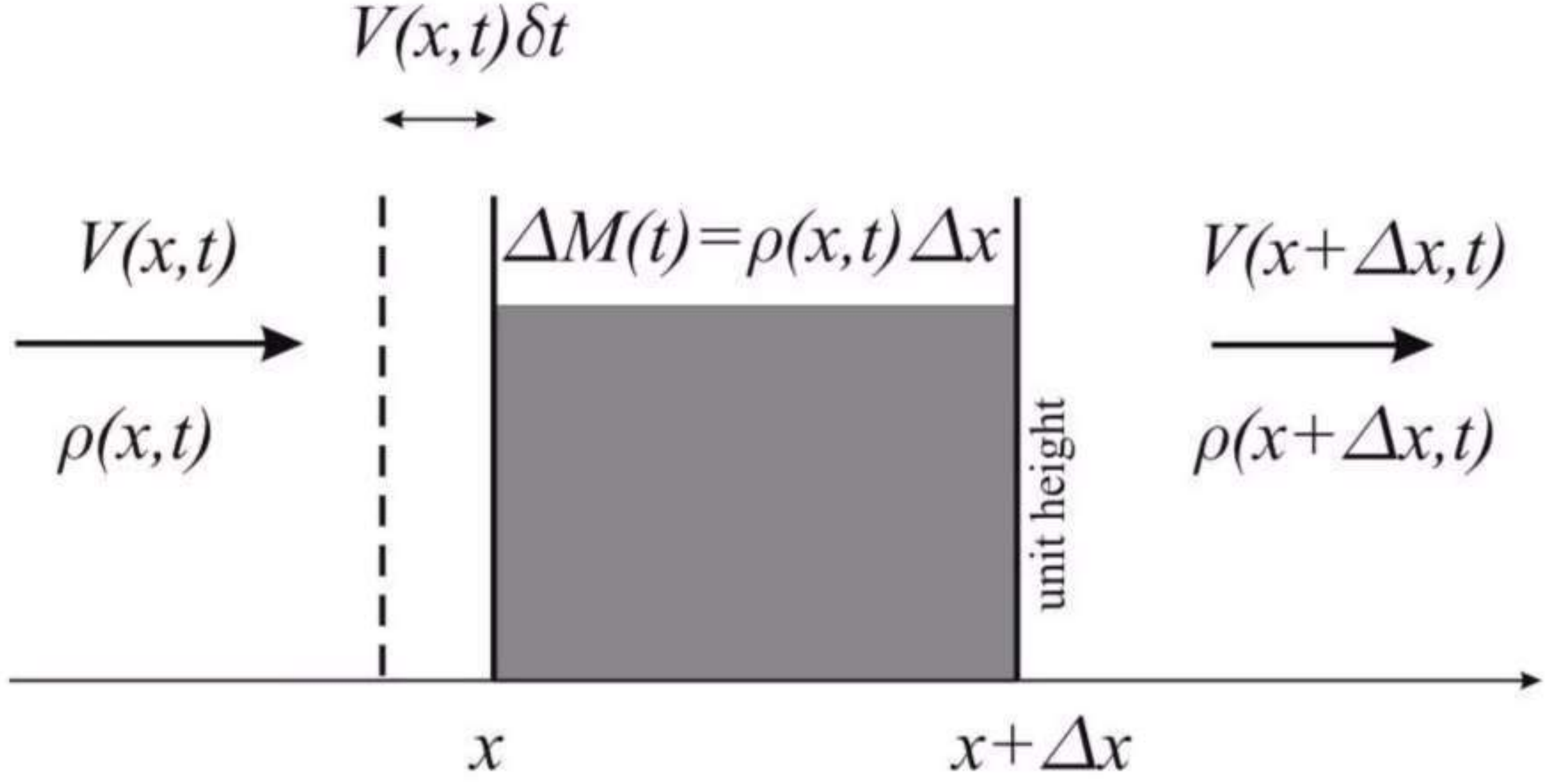
2D-example:

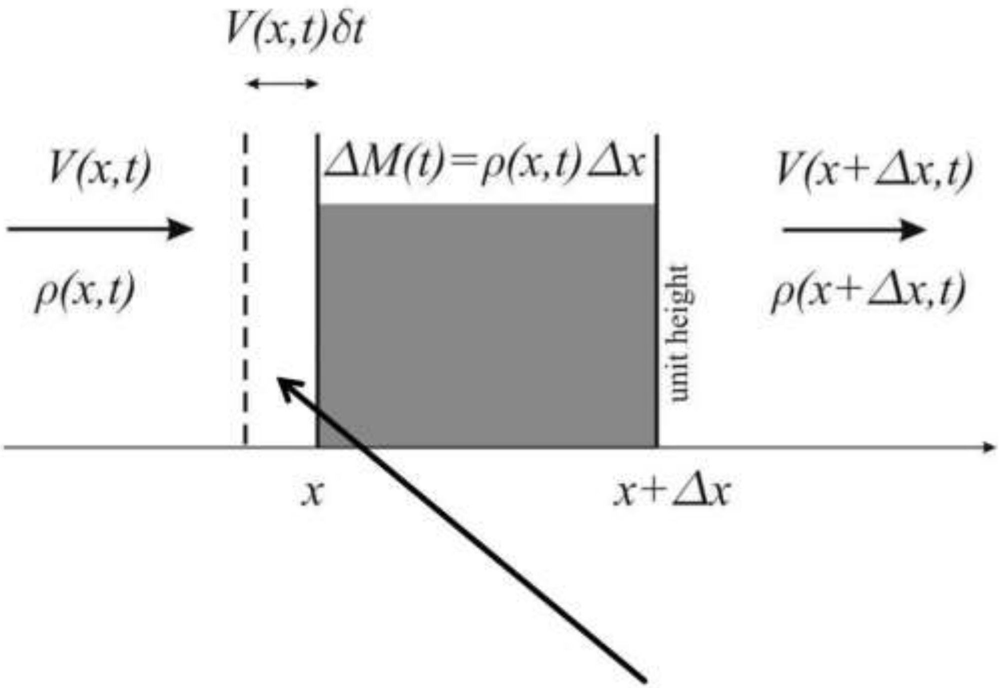
A fluid filament is deformed and stretched by the flow;

Its area changes, but the mass contained in the filament can NOT change

<u>So:</u> the mass density must change in response to the flow!

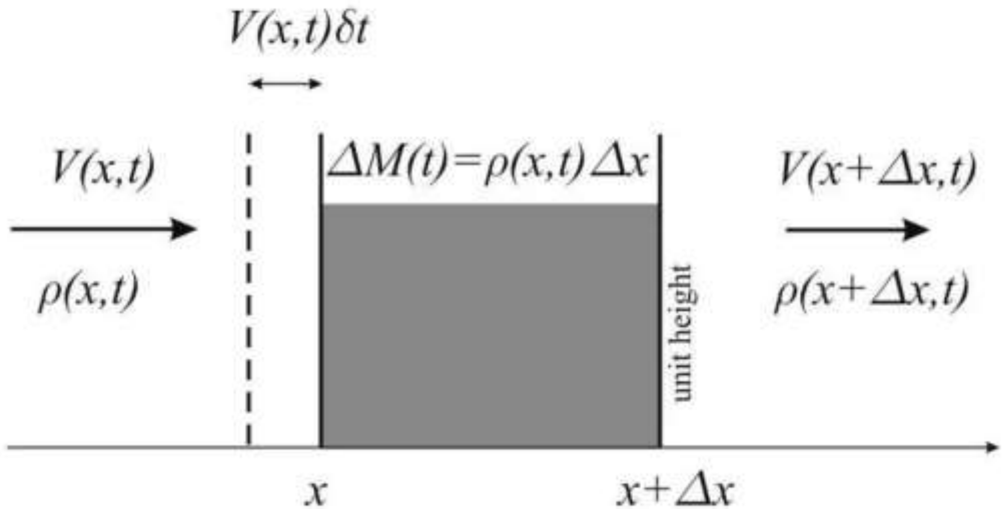
Simple one-dimensional flow:





left boundary box: $\Delta M_{\text{in}} = \rho(x,t)V(x,t)\delta t$

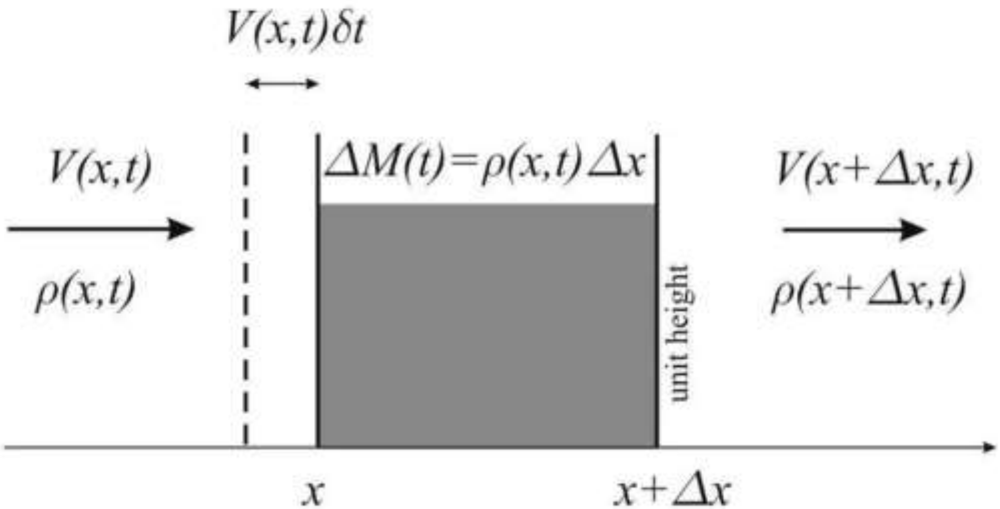
right boundary box: $\Delta M_{\text{out}} = \rho(x+\Delta x,t)V(x+\Delta x,t)\delta t$



$$\delta\left(\Delta M\right) \equiv \frac{\mathrm{d}\left(\Delta M\right)}{\mathrm{d}t}\delta t = \Delta M_{\text{in}} - \Delta M_{\text{out}}$$

$$= \rho(x,t)V(x,t)\delta t - \rho(x+\Delta x,t)V(x+\Delta x,t)\delta t$$

$$\approx -\delta t\ \Delta x \frac{\partial}{\partial x}\left(\rho V\right)$$

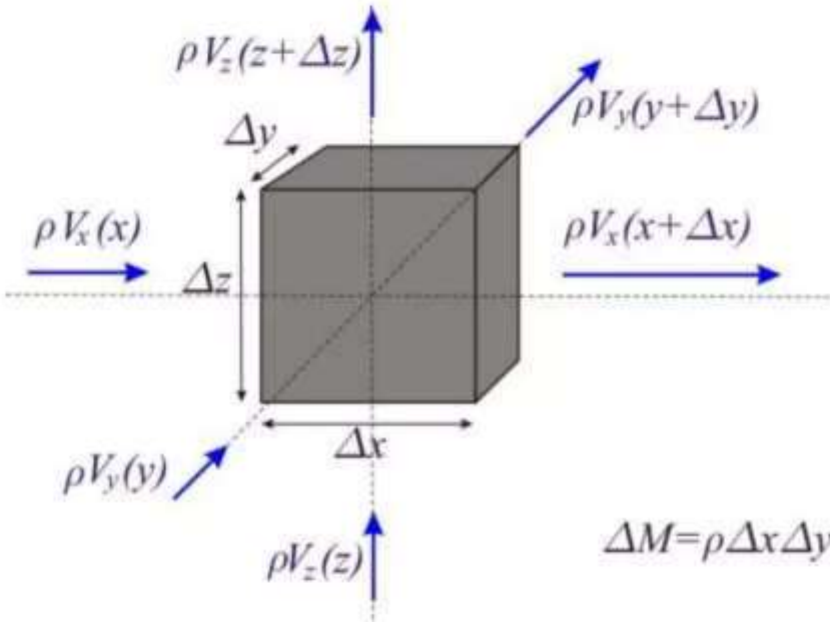


$$\frac{\mathrm{d}(\Delta M)}{\mathrm{d}t}\delta t = \delta t\ \Delta x\left(\frac{\partial \rho}{\partial t}\right) = -\delta t\ \Delta x\frac{\partial}{\partial x}(\rho V)$$

$$\Leftrightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0$$

Generalization to three dimensions:

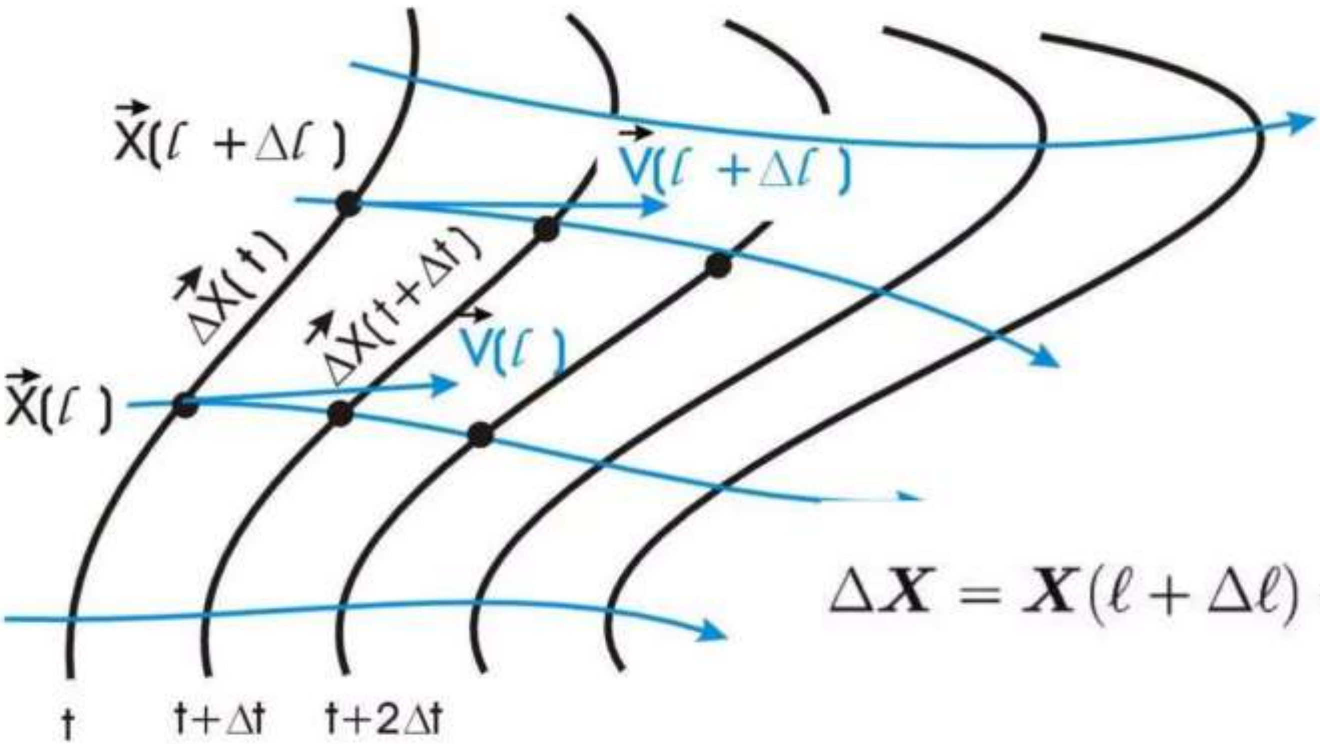


$\Delta M = \rho \Delta x \Delta y \Delta z$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho V_x\right) + \frac{\partial}{\partial y}\left(\rho V_y\right) + \frac{\partial}{\partial z}\left(\rho V_z\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \tilde{\mathbf{N}} \bullet \left(\rho V\right) = 0$$

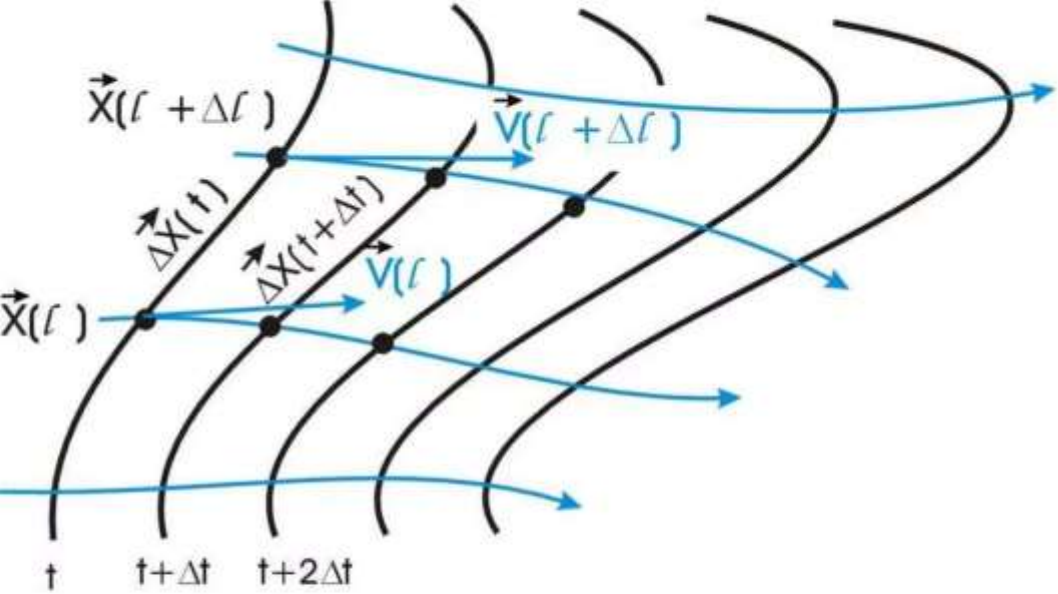
Curves, tangent vectors and volumes carried by flow



$$\Delta \boldsymbol{X} = \boldsymbol{X}(\ell + \Delta\ell) - \boldsymbol{X}(\ell) \approx \frac{\partial \boldsymbol{X}}{\partial \ell}\,\Delta\ell$$

Curve carried by flow

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{V}(\boldsymbol{x} = \boldsymbol{X}, t)$$



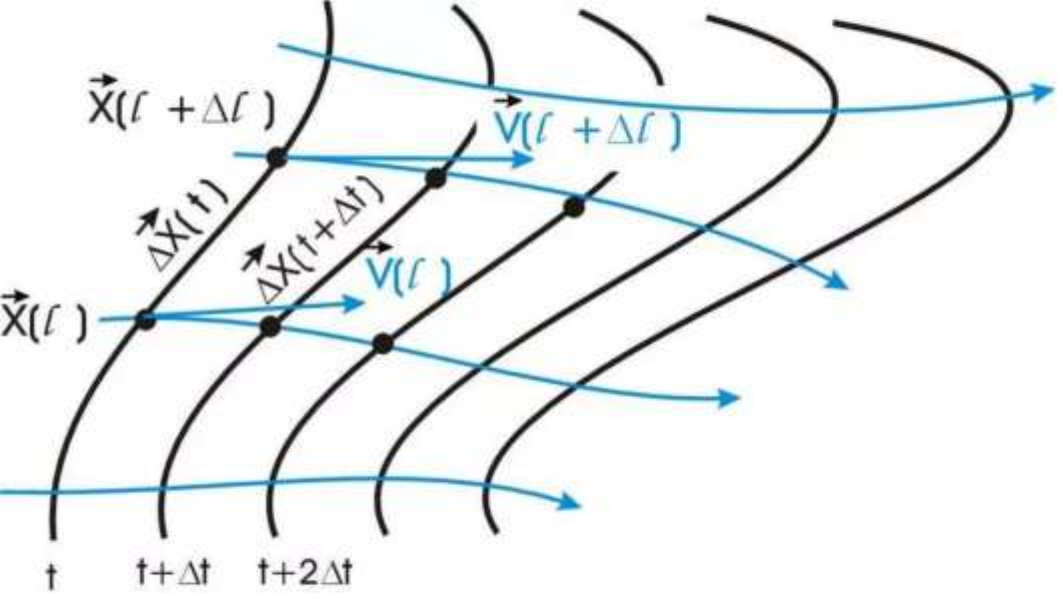
Curve carried by flow

Velocity at each point equals fluid velocity:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{V}(\boldsymbol{x} = \boldsymbol{X}, t)$$

$$\Delta \boldsymbol{X} = \boldsymbol{X}(\ell + \Delta\ell) - \boldsymbol{X}(\ell) \approx \frac{\partial \boldsymbol{X}}{\partial \ell}\,\Delta\ell$$

Definition of tangent vector



Curve carried by flow

Velocity at each point equals fluid velocity:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{V}(\boldsymbol{x} = \boldsymbol{X}, t)$$

Definition of tangent vector:

$$\Delta\boldsymbol{X} = \boldsymbol{X}(\ell + \Delta\ell) - \boldsymbol{X}(\ell) \approx \frac{\partial \boldsymbol{X}}{\partial \ell}\,\Delta\ell$$

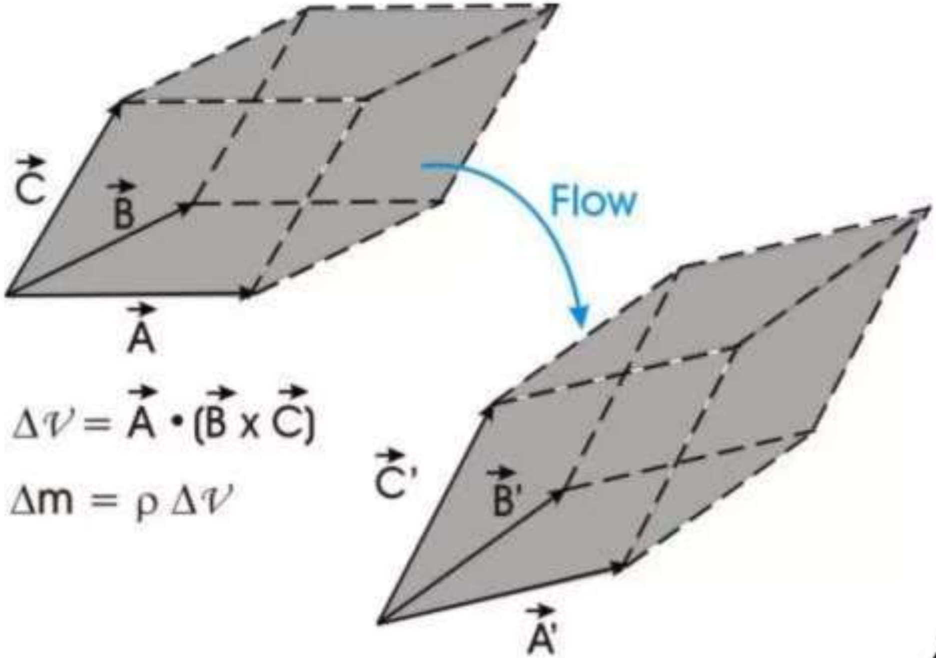
Equation of motion of tangent vector:

$$\frac{\mathrm{d}(\Delta\boldsymbol{X})}{\mathrm{d}t} = \boldsymbol{V}(\boldsymbol{X}(\ell) + \Delta\boldsymbol{X}, t) - \boldsymbol{V}(\boldsymbol{X}(\ell), t)$$

$$\approx (\Delta\boldsymbol{X}\cdot\boldsymbol{\nabla})\boldsymbol{V}$$

Volume: definition

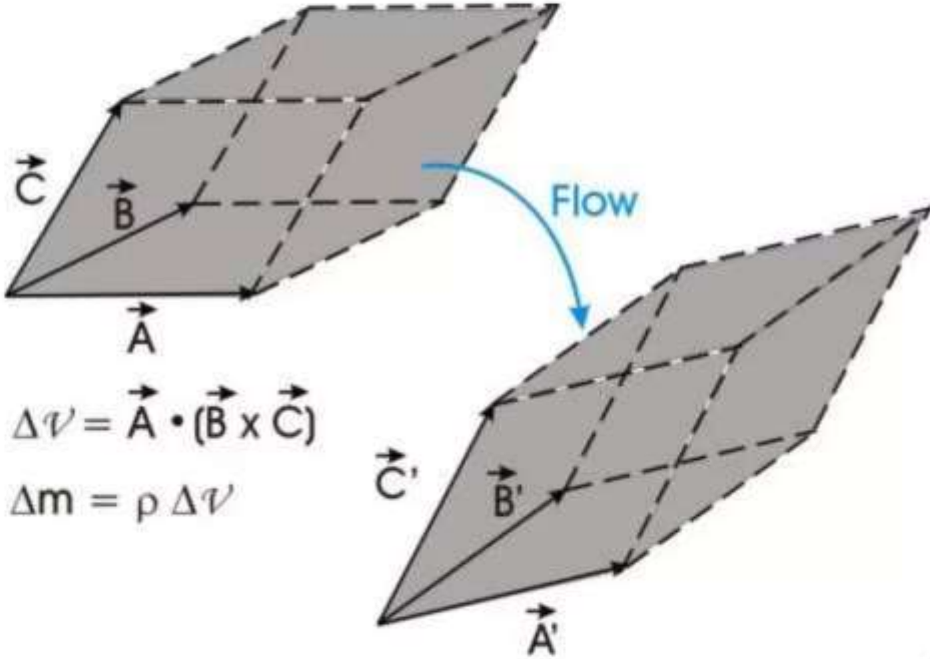
$$\overrightarrow{A} = \overrightarrow{\Delta X}, \overrightarrow{B} = \overrightarrow{\Delta Y}, \overrightarrow{C} = \overrightarrow{\Delta Z}$$



$$\Delta\mathcal{V} = \Delta\boldsymbol{X} \cdot (\Delta\boldsymbol{Y} \times \Delta\boldsymbol{Z}) = \begin{Vmatrix} \Delta X_x & \Delta X_y & \Delta X_z \\ \Delta Y_x & \Delta Y_y & \Delta Y_z \\ \Delta Z_x & \Delta Z_y & \Delta Z_z \end{Vmatrix}$$

The vectors A, B and C are carried along by the flow!

Volume: definition



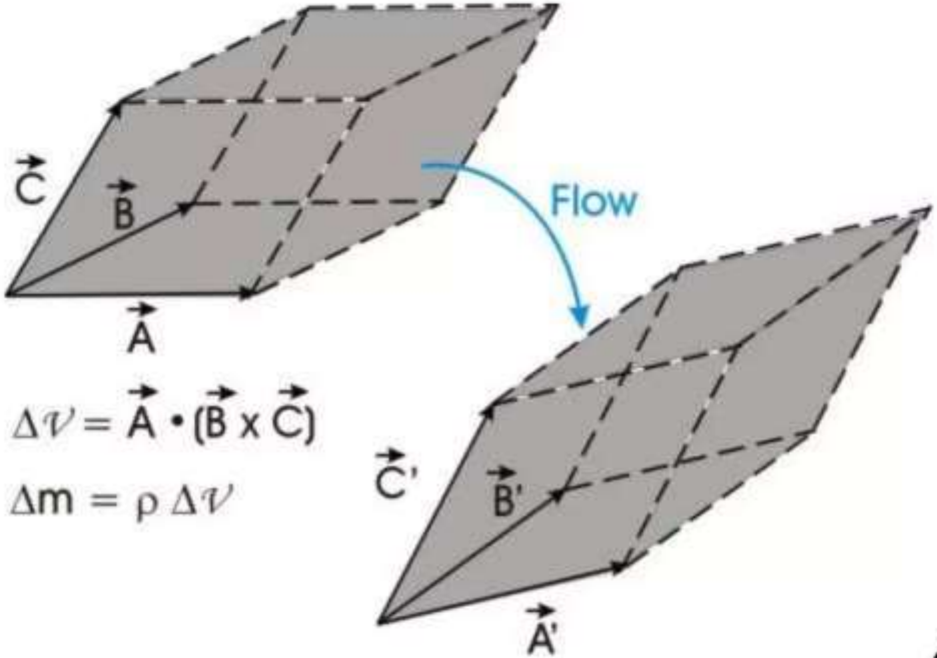
$$\overrightarrow{A} = \overrightarrow{\Delta X}\ ,\ \overrightarrow{B} = \overrightarrow{\Delta Y},\ \overrightarrow{C} = \overrightarrow{\Delta Z}$$

$$\Delta\mathcal{V} = \Delta\boldsymbol{X}\cdot(\Delta\boldsymbol{Y}\times\Delta\boldsymbol{Z}) = \begin{Vmatrix} \Delta X_x & \Delta X_y & \Delta X_z \\ \Delta Y_x & \Delta Y_y & \Delta Y_z \\ \Delta Z_x & \Delta Z_y & \Delta Z_z \end{Vmatrix}$$

$$\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t} = \frac{\mathrm{d}\Delta\boldsymbol{X}}{\mathrm{d}t}\cdot(\Delta\boldsymbol{Y}\times\Delta\boldsymbol{Z})$$

$$+\Delta\boldsymbol{X}\cdot\left(\frac{\mathrm{d}\Delta\boldsymbol{Y}}{\mathrm{d}t}\times\Delta\boldsymbol{Z}+\Delta\boldsymbol{Y}\times\frac{\mathrm{d}\Delta\boldsymbol{Z}}{\mathrm{d}t}\right)$$

Volume: definition



$$\overrightarrow{A} = \overrightarrow{\Delta X},\ \overrightarrow{B} = \overrightarrow{\Delta Y},\ \overrightarrow{C} = \overrightarrow{\Delta Z}$$

$$\Delta\mathcal{V} = \Delta\boldsymbol{X}\cdot(\Delta\boldsymbol{Y}\times\Delta\boldsymbol{Z}) = \begin{Vmatrix} \Delta X_x & \Delta X_y & \Delta X_z \\ \Delta Y_x & \Delta Y_y & \Delta Y_z \\ \Delta Z_x & \Delta Z_y & \Delta Z_z \end{Vmatrix}$$

$$\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t} = \frac{\mathrm{d}\Delta\boldsymbol{X}}{\mathrm{d}t}\cdot(\Delta\boldsymbol{Y}\times\Delta\boldsymbol{Z}) + \Delta\boldsymbol{X}\cdot\left(\frac{\mathrm{d}\Delta\boldsymbol{Y}}{\mathrm{d}t}\times\Delta\boldsymbol{Z} + \Delta\boldsymbol{Y}\times\frac{\mathrm{d}\Delta\boldsymbol{Z}}{\mathrm{d}t}\right)$$

$$\frac{\mathrm{d}(\Delta\boldsymbol{X})}{\mathrm{d}t} = (\Delta\boldsymbol{X}\cdot\boldsymbol{\nabla})\boldsymbol{V}$$

$$\frac{\mathrm{d}(\Delta\boldsymbol{Y})}{\mathrm{d}t} = (\Delta\boldsymbol{Y}\cdot\boldsymbol{\nabla})\boldsymbol{V}$$

$$\frac{\mathrm{d}(\Delta\boldsymbol{Z})}{\mathrm{d}t} = (\Delta\boldsymbol{Z}\cdot\boldsymbol{\nabla})\boldsymbol{V}$$

$$\Delta \boldsymbol{X} = \begin{pmatrix} \Delta X \\ 0 \\ 0 \end{pmatrix}, \ \Delta \boldsymbol{Y} = \begin{pmatrix} 0 \\ \Delta Y \\ 0 \end{pmatrix}, \ \Delta \boldsymbol{Z} = \begin{pmatrix} 0 \\ 0 \\ \Delta Z \end{pmatrix}$$

Special choice: orthogonal triad

$$\Delta \mathcal{V} = \Delta X \, \Delta Y \, \Delta Z$$

$$\frac{\mathrm{d}\Delta \mathcal{V}}{\mathrm{d}t} = \boxed{\frac{\mathrm{d}\Delta \boldsymbol{X}}{\mathrm{d}t} \cdot (\Delta \boldsymbol{Y} \times \Delta \boldsymbol{Z})} + \Delta \boldsymbol{X} \cdot \left(\frac{\mathrm{d}\Delta \boldsymbol{Y}}{\mathrm{d}t} \times \Delta \boldsymbol{Z} + \Delta \boldsymbol{Y} \times \frac{\mathrm{d}\Delta \boldsymbol{Z}}{\mathrm{d}t}\right)$$

General volume-change law

$$\Delta \boldsymbol{X} = \begin{pmatrix} \Delta X \\ 0 \\ 0 \end{pmatrix}, \ \Delta \boldsymbol{Y} = \begin{pmatrix} 0 \\ \Delta Y \\ 0 \end{pmatrix}, \ \Delta \boldsymbol{Z} = \begin{pmatrix} 0 \\ 0 \\ \Delta Z \end{pmatrix}$$

Special choice: Orthonormal triad

$$\Delta \mathcal{V} = \Delta X \, \Delta Y \, \Delta Z$$

$$\frac{\mathrm{d}\Delta \mathcal{V}}{\mathrm{d}t} = \boxed{\frac{\mathrm{d}\Delta \boldsymbol{X}}{\mathrm{d}t} \cdot (\Delta \boldsymbol{Y} \times \Delta \boldsymbol{Z})} + \Delta \boldsymbol{X} \cdot \left(\frac{\mathrm{d}\Delta \boldsymbol{Y}}{\mathrm{d}t} \times \Delta \boldsymbol{Z} + \Delta \boldsymbol{Y} \times \frac{\mathrm{d}\Delta \boldsymbol{Z}}{\mathrm{d}t} \right)$$

General Volume-change law

$$\Delta X \left\| \begin{matrix} \partial V_x/\partial x & \partial V_y/\partial x & \partial V_z/\partial x \\ 0 & \Delta Y & 0 \\ 0 & 0 & \Delta Z \end{matrix} \right\| = \left(\frac{\partial V_x}{\partial x} \right) \underbrace{\Delta X \, \Delta Y \, \Delta Z}_{\text{volume} \Delta \mathcal{V}}$$

Mass conservation and the continuity equation

$$\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t} = \Delta X \Delta Y \Delta Z \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) = \Delta\mathcal{V}\left(\boldsymbol{\nabla}\cdot\boldsymbol{V}\right)$$

Volume change

Mass conservation: $\rho\,\Delta V$ = constant

$$\frac{\mathrm{d}(\rho\Delta\mathcal{V})}{\mathrm{d}t} = \Delta\mathcal{V}\,\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t} = 0$$

Mass conservation and the continuity equation

$$\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t} = \Delta X \Delta Y \Delta Z \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) = \Delta\mathcal{V}\left(\boldsymbol{\nabla}\cdot\boldsymbol{V}\right)$$

Volume change

Comoving derivative

Mass conservation: $\rho\,\Delta V$ = constant

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})$$

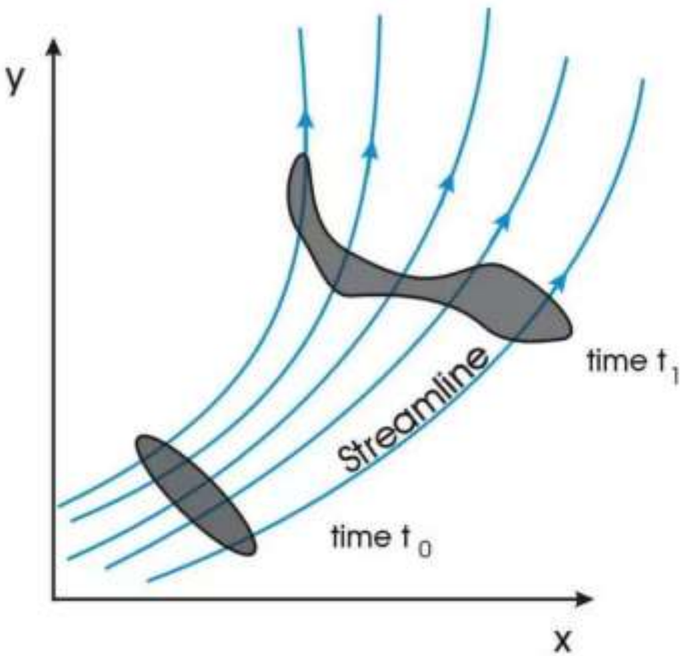
$$\frac{\mathrm{d}(\rho\Delta\mathcal{V})}{\mathrm{d}t} = \Delta\mathcal{V}\,\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial\rho}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho = -\rho\left(\frac{1}{\Delta\mathcal{V}}\,\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t}\right) = -\rho(\boldsymbol{\nabla}\cdot\boldsymbol{V})$$

The continuity equation : the behaviour of the mass-density

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho = -\rho\left(\frac{1}{\Delta\mathcal{V}}\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t}\right) = -\rho(\boldsymbol{\nabla} \cdot \boldsymbol{V})$$

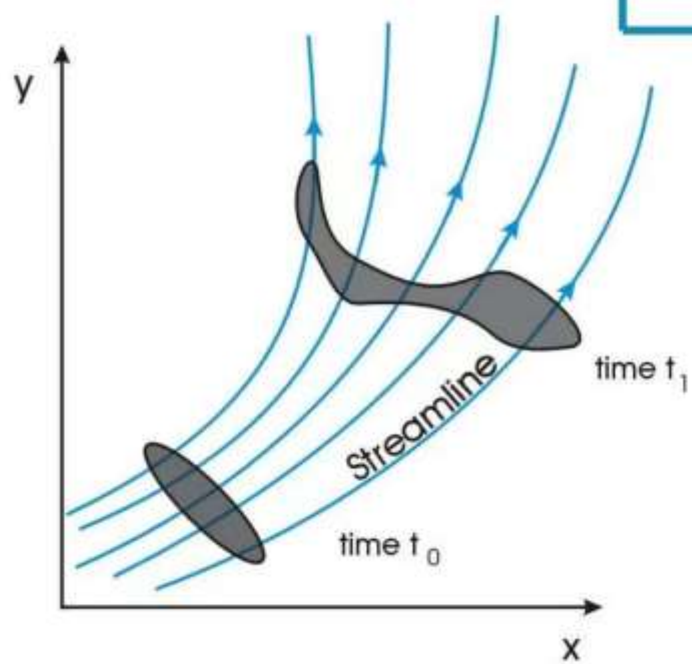
$$\frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho + \rho\,(\boldsymbol{\nabla} \cdot \boldsymbol{V}) = 0$$



The continuity equation : the behaviour of the mass-density

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho = -\rho\left(\frac{1}{\Delta\mathcal{V}}\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t}\right) = -\rho(\boldsymbol{\nabla} \cdot \boldsymbol{V})$$

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho + \rho\,(\boldsymbol{\nabla} \cdot \boldsymbol{V}) = 0$$



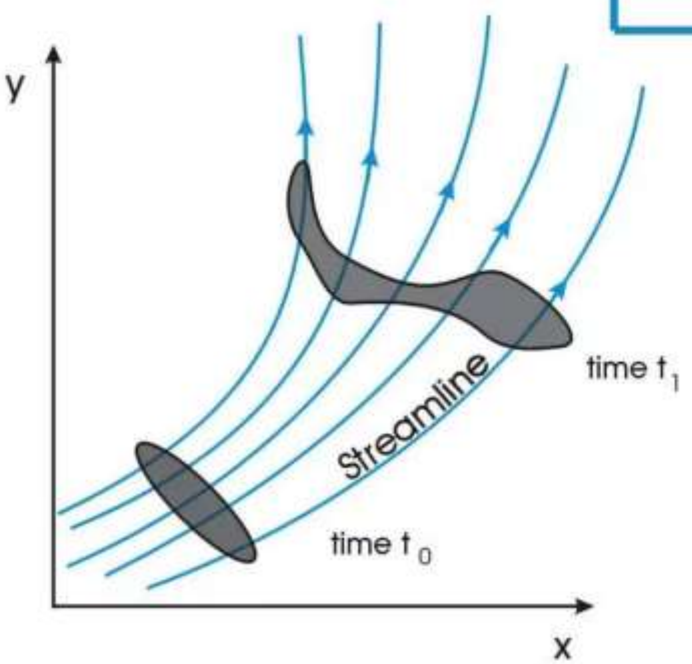
$$\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{A} \cdot \boldsymbol{\nabla})f$$

Divergence product rule

The continuity equation : the behaviour of the mass-density

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho = -\rho\left(\frac{1}{\Delta\mathcal{V}}\frac{\mathrm{d}\Delta\mathcal{V}}{\mathrm{d}t}\right) = -\rho(\boldsymbol{\nabla} \cdot \boldsymbol{V})$$

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho + \rho\,(\boldsymbol{\nabla} \cdot \boldsymbol{V}) = 0$$



$$\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = f(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{A} \cdot \boldsymbol{\nabla})f$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\,\boldsymbol{V}) = 0$$

Summary: we are almost there!

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})$$

$$\rho \left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right) = -\boldsymbol{\nabla} P + \text{other (external) forces}$$

$$P(\rho\,,\,T) = n k_{\mathrm{b}} T = \frac{\rho \mathcal{R} T}{\mu} \qquad \& \qquad P\,\rho^{-5/3} = \text{constant}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\,\boldsymbol{V}) = 0$$